SHEAF COHOMOLOGY COURSE NOTES, SPRING 2010

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Contents

Overview

The basic plan is to follow [H, chapter III], filling in the details where he refers to other books. (1/25/10)

Planned additional topics (some more ambitious than others):

(1) Using Frobenius splitting [BrKu05] to show the vanishing of some sheaf cohomology groups.
(2) Spectral sequences; e.g. the spectral sequence associated to a filtration of a complex.

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(3) Local cohomology, and its relation to depth, normality, and Cohen-Macaulayness.
(4) Atiyah and Bott's marriage of the Riemann-Roch and Lefschetz theorems.

1. *My* MOTIVATION: K-THEORY OF SCHEMES

(1/25/10) Let X be a scheme, and define the group $K_0(X)$[1] as the free abelian group on the set of isomorphism classes $\{[\mathcal{F}]\}$ of coherent sheaves on X, "modulo exact sequences", which means that for any finite exact sequence

$$0 \to \mathcal{F}_1 \to \mathcal{F}_1 \to \cdots \to \mathcal{F}_m \to 0$$

we mod out by $[\mathcal{F}_1] - [\mathcal{F}_2] + \ldots - (-1)^m[\mathcal{F}_m]$.

Exercise 1.1. Why is this a set, rather than a proper class?

Exercise 1.2. What if we had ignored set-theoretic difficulties, but had taken the free abelian group on the class of coherent sheaves, rather than isomorphism classes thereof?

Exercise 1.3. What if we only mod out by short exact sequences?

(1/25/10) **Exercise** 1.4. Compute $K_0(X)$ for $X = \mathrm{Spec}\,(\mathbf{k})$.

Exercise 1.5. Compute $K_0(X)$ for $X = \mathrm{Spec}\,(\mathbf{k}[x])$.

(1/25/10) If X is projective over a field $\mathbf{k}$, then the space $\Gamma(X;\mathcal{F})$ of global sections of a coherent sheaf $\mathcal{F}$ is finite-dimensional [H, theorem 5.19]. That suggests that we might have a homomorphism

$$K_0(X) \to \mathbb{Z}, \quad [\mathcal{F}] \to \dim \Gamma(X;\mathcal{F})$$

but this doesn't work, because $\Gamma(X;\bullet)$ *does not take exact sequences to exact sequences*. All we get is **left exactness**:

(1/25/10)

Theorem. [H, exercise II.1.8] *If* $0 \to \mathcal{F}_1 \xrightarrow{\alpha} \mathcal{F}_2 \xrightarrow{\beta} \mathcal{F}_3$ *is exact, then* $0 \to \Gamma(X;\mathcal{F}_1) \xrightarrow{\alpha'} \Gamma(X;\mathcal{F}_2) \xrightarrow{\beta'} \Gamma(X;\mathcal{F}_3)$ *is exact too.*

Proof. First we need to show α' is injective. Take $s \in \ker \alpha' \leq \Gamma(X;\mathcal{F}_1)$, so $\alpha'(s) = 0 \in \Gamma(X;\mathcal{F}_2)$. Since α is a sheaf homomorphism, it is determined by its induced maps on stalks. Since $\alpha'(s)$ is 0 on each stalk, $s \in \ker\alpha$ on each stalk, so s is zero on each stalk, so s is zero.

Now take $s \in \ker \beta' \leq \Gamma(X;\mathcal{F}_2)$. The assumption that the complex of sheaves is exact at $\mathcal{F}_2$ says that for each point $x \in X$, there is an open set $U_x \ni x$ and a section $r_x \in \Gamma(U_x;\mathcal{F}_1)$ such that $s = \alpha'(r_x)$ on U_x.

To glue these $\{r_x\}$ together, we need to know they agree on their common open sets of definition. Given x, y look at $r_x - r_y$ on $U_x \cap U_y$. Then $\alpha'(r_x) = s = \alpha'(r_y)$ on U_x, so $\alpha'(r_x - r_y) = 0$, so $r_x - r_y = 0$ (which is where we make use of the "$0 \to$" on the left of the sheaf exact sequence). Hence the $\{r_x\}$ glue together to a section $r \in \Gamma(X;\mathcal{F}_1)$ with $\alpha'(r) = s$, as was to be shown. ∎

(1/25/10) **Exercise** 1.6. Let $X = \mathbb{CP}^1$, and T^*X denote its cotangent sheaf. Show that $\Gamma(X;T^*X) = 0$. Hint: show $\Gamma(X;TX) \neq 0$ first.

[1] Sometimes called $G(X)$, but we'll reserve G for a group. There are very complicated groups $K_{i>0}(X)$ that we won't deal with at all; people can't even compute them for $X = \mathrm{Spec}\,(\text{field})$.

(1/25/10) **Exercise** 1.7. Let $X = \mathbb{CP}^1$, and $\vec{v} \in \Gamma(X; TX)$ a vector field rotating $\mathbb{CP}^1$ fixing the poles $\{0, \infty\}$. Consider the sequence

$$0 \to T^*X \to \mathcal{O}_X \to \mathcal{O}_{\{0,\infty\}} \to 0$$

taking 1-forms to functions to functions at the poles, where the first map is "dot with $\vec{v}$" and the second is restriction. Show this is an exact sequence of sheaves, but that $\Gamma(X; \bullet)$ of it is not short exact.

(1/25/10) There is in fact a homomorphism $K_0(X) \to \mathbb{Z}$, but it takes $[\mathcal{F}] \mapsto \dim \Gamma(X; \mathcal{F}) \pm$ correction terms. More generally, if $\phi : X \to Y$ is a projective morphism, then there is a group homomorphism $K_0(X) \to K_0(Y)$ taking $[\mathcal{F}] \mapsto [\phi_!(\mathcal{F})] \pm$ correction terms.

(1/25/10) Long-term goals for K-theory in this class:

(1) Define these correction terms.
(2) Give ways to compute them (one will be: fit them into a long exact sequence).
(3) Give criteria to show that they vanish (so the naïve guess $[\mathcal{F}] \mapsto [\phi_!(\mathcal{F})]$ works).
(4) Show that $K_0(\bullet)$ is a homology theory.
(5) Define a corresponding cohomology theory $K^0(\bullet)$, and prove a Poincaré duality for X regular and complete.

(1/25/10) Shorter term, we'll look at a simpler (really, more affine) situation with most of the same features, where the global sections functor $\Gamma(X; \bullet)$ is replaced by $\mathrm{Hom}(M, \bullet)$, $\mathrm{Hom}(\bullet, M)$ and $\otimes M$ functors.

2. First steps in homological algebra

Some of this is from [GM] rather than [H].

Example. Let ${}_RM_S$ be an (R, S)-bimodule. Then there are additive functors

$$M \otimes_S \bullet : S - \mathbf{Mod} \to R - \mathbf{Mod}, \qquad \mathrm{Hom}_R(\bullet, M) : R - \mathbf{Mod} \to S^{op} - \mathbf{Mod}$$

covariant and contravariant, respectively.

(1/27/10) **Definition. Complexes** and **exact sequences** (A_i) of modules. The **cohomology** of a complex (our differentials will always have degree $+1$). Categories of complexes of modules.

As with $\Gamma(X; \bullet)$ in the last section, the contravariant functor $\mathrm{Hom}_R(\bullet, M)$ does not take short exact sequences

$$0 \to A \to B \to C \to 0$$

to short exact sequences

$$0 \to \mathrm{Hom}(C, M) \to \mathrm{Hom}(B, M) \to \mathrm{Hom}(A, M) \overset{\text{not}}{\to} 0;$$

one must remove the leading 0. Hom is only **left exact**.

(1/27/10) **Exercise** 2.1. Prove the left exactness.

Exercise 2.2. Give an example where A, B, C, M are finite abelian groups (hence $\mathbb{Z}$-modules) where right exactness fails.

2.1. **What replaces right exactness?** (1/27/10) Idea #1: maybe the definition of $\mathrm{Hom}(A, M)$ is "wrong"; it needs correction terms. What could come next? Of course we could just put the cokernel there, but that turns out to be hard to compute.

Perhaps instead of just studying maps $A \to M$, we should next study chains $A \to E \to M$. The most basic is short exact sequences $0 \to A \xrightarrow{i} E \xrightarrow{\pi} M \to 0$.

Definition. Let $0 \to A \xrightarrow{i_1} E_1 \xrightarrow{\pi_1} M \to 0$, $0 \to A \xrightarrow{i_2} E_2 \xrightarrow{\pi_2} M \to 0$ be two short exact sequences. Define

$$E = (E_1 \times_M E_2)/\{(i_1(a), -i_2(a)) : a \in A\}$$

and observe that there is another natural exact sequence $0 \to A \to E \to M \to 0$. Let

$$\begin{aligned} \mathrm{Ext}_1(A, M) &:= (\text{the free } \mathbb{Z}\text{-module on isomorphism classes } [A \to E \to M]) \\ &\qquad \Big/\Big\langle [A \to E_1 \to M] + [A \to E_2 \to M] = [A \to E \to M] \Big\rangle. \end{aligned}$$

Exercise 2.2.1. Confirm that $0 \to A \to E \to M \to 0$ is indeed exact.

Exercise 2.2.2. Show that $[0 \to A \to A \oplus M \to M \to 0]$ is the identity in this group.

Exercise 2.2.3. If R is a field, show that $\mathrm{Ext}_1(A, M)$ vanishes.

Exercise 2.2.4. Let $0 \to A \to B \to C \to 0$ be exact. Define an exact sequence

$$\mathrm{Hom}(B, M) \to \mathrm{Hom}(C, M) \to \mathrm{Ext}_1(A, M) \to \mathrm{Ext}_1(B, M) \to \mathrm{Ext}_1(C, M).$$

Exercise 2.2.5. If $R = \mathbb{Z}$, show that the last map in 2.2.4 is onto.

2.2. **How to extend further?** We could probably pursure longer sequences $A \to \cdots \to M$, but we take a different tack.

(1/27/10) Maybe only for particularly good A or M is $\mathrm{Hom}(A, M)$ already the "right" definition. It is easy to define goodness:

(1/27/10) **Definition.** A left R-module A is **projective** if $\mathrm{Hom}_R(A, \bullet)$ is left exact. (This "projective" is more related to projecting, rather than to projective space.)

Exercise 2.3. Show free modules are projective.

(1/27/10)

Theorem 1. *TFAE:*

1. *A is projective.*
2. *Any surjection $M \twoheadrightarrow A$ is a projection, i.e. $M \cong A \oplus A'$. (This seems the best motivation for the name.)*
3. *A is a direct summand of a free module.*
4. *M has the "lifting property" that given any $N \twoheadrightarrow M \leftarrow A$, one can factor the $A \to M$ map through N as $A \to N \twoheadrightarrow M$.*

Proof. (1) $\implies$ (2). Define $0 \to K \to M \xrightarrow{\pi} A \to 0$ using the surjection. Then by (1), the associated

$$0 \to \mathrm{Hom}(A, K) \to \mathrm{Hom}(A, M) \to \mathrm{Hom}(A, A) \to 0$$

is also exact. Pick $\tau \in \mathrm{Hom}(A, M)$ such that $\tau \mapsto 1_A$. Then $\pi \circ \tau = 1_A$, so the image of τ defines the copy of A complementary to $K = A'$.

(2) $\implies$ (3). A is the image of a free module M, hence is a direct summand by (2).

(3) $\implies$ (4). Extend the $A \to M$ map to a $A \oplus A' \to M$ map (e.g. by 0 on A'). For each chosen generator of this free module, lift from M to a preimage in N (which exists by ontoness). Then restrict the map so defined back to $A \leq A \oplus A'$.

(4) $\implies$ (1). **Exercise** 2.4. □

(1/27/10)

Now, why would a nice short list of modules produced a long sequence?

(1/27/10) Idea #2: maybe the input (A, M) to $\mathrm{Hom}(\bullet, \bullet)$ *shouldn't* be a pair of modules, but a pair of complexes, and only for very simple complexes is the usual definition "right". We now pursue those complexes:

(1/27/10) **Definition.** A **projective resolution** of a module A is a complex

$$\cdots \to P_{-n} \to P_{-n+1} \to \cdots \to P_0 \to 0 \to \cdots$$

of projective modules, exact away from degree 0, where its cohomology is A. [2]

Exercise 2.6. Show that every A possesses a projective resolution.

(1/27/10) **Definition.** If (P_i) is a projective resolution of A, the associated complex

$$\cdots \to 0 \to \mathrm{Hom}(P_0, M) \to \mathrm{Hom}(P_{-1}, M) \to \dots$$

may not be exact: define $\mathrm{Ext}_i(A, M)$ to be its ith cohomology group.

Exercise 2.7. $\mathrm{Ext}_0(A, M) = \mathrm{Hom}(A, M)$.

Exercise 2.8. Show that the two definitions of $\mathrm{Ext}_1(A, M)$ are naturally isomorphic.

Exercise 2.9. Show $\mathrm{Ext}_1(A, M)$ vanishes for modules over a field.

Exercise 2.10. Give A, M modules over $\mathbb{Z}$ s.t. $\mathrm{Ext}_1(A, M) \neq 0$.

Exercise 2.11. Show $\mathrm{Ext}_2(A, M)$ vanishes for f.g. $\mathbb{Z}$-modules.

What about uniqueness? We need a way to compare two complexes.

Definition. A **chain homotopy** $h : (A_i) \to (B_i)$ between two chain maps $f_0, f_1 : (A_i) \to (B_i)$ is a degree -1 map such that

$$d_B \circ h + h \circ d_A = f_0 - f_1.$$

("Why" *anti*commutator? "Because" both d and h are odd-degree operators.) Maps $f : (A_i) \to (B_i)$, $g : (B_i) \to (A_i)$ are **homotopy inverses** if $f \circ g$, $g \circ f$ have chain homotopies to the identity.

Proposition 1. (1) *Functors take morphisms of complexes to morphisms of complexes, and homotopies to homotopies.*
(2) *Homotopy inverses induce isomorphisms on cohomology.*
(3) *Functors take homotopic complexes to homotopic complexes.*

Here's the real reason to use projective resolutions:

Theorem 2. *Let $(P_\bullet), (Q_\bullet)$ be two projective resolutions of A. Then their complexes are homotopic.*

[2] Alternately, one may insist that $P_0 = A$ and that the complex be exact everywhere; it is easy to pass between the two definitions. Ours is the one that fits better with the "derived category" philosophy. Obviously people like to index by $\mathbb{N}$ rather than $-\mathbb{N}$, too.

Proof. We use (Q_i) projective to construct a morphism Φ of exact complexes

$$\begin{array}{ccccccccc} \cdots & P_{-2} & \to & P_{-1} & \to & P_0 & \to & A & \to 0 \\ & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\ \cdots & Q_{-2} & \to & Q_{-1} & \to & Q_0 & \to & A & \to 0 \end{array}$$

and similarly construct Ψ in the opposite direction. Throwing away the A to obtain morphisms of the resolutions, we see Φ, Ψ induce inverse isomorphisms on $H^*(P), H^*(Q)$, both supported in degree 0.

Now consider the composite $\beta := \Phi \circ \Psi$ as a chain endomorphism $(P_i) \to (P_i)$, inducing the identity on $H^0((P_i)) = A$. Then $1 - \beta$ induces the zero map on cohomology, so maps every P_i (including P_0) into the kernel of $P_i \to P_{i+1}$. Since $P_{i-1} \twoheadrightarrow \ker(P_i \to P_{i+1})$, and P_i is projective, we can pick $h_i : P_i \to P_{i-1}$. The rest is routine ☐

3. The long exact sequence

(1/27/10) Where do long exact sequences come from?

(1/27/10)

Proposition 2. *Let $0 \to (A_i) \to (B_i) \to (C_i) \to 0$ be a short exact sequence of complexes. Then there is a naturally associated long exact sequence on cohomology,*

$$\cdots \to H^i(A) \to H^i(B) \to H^i(C) \to H^{i+1}(A) \to \cdots$$

Proof. Diagram chase. ☐

(1/29/10) So now we would like to start with a short exact sequence of modules,

$$\begin{array}{c} 0 \\ \downarrow \\ A \\ \downarrow \\ B \\ \downarrow \\ C \\ \downarrow \\ 0, \end{array}$$

replace each by a projective resolution, and get a short exact sequence

$$\begin{array}{ccccc}
 & 0 & & 0 & \\
 & \downarrow & & \downarrow & \\
\cdots \to & A_1 & \to & A_0 & \to 0 \\
 & \downarrow & & \downarrow & \\
\cdots \to & B_1 & \to & B_0 & \to 0 \\
 & \downarrow & & \downarrow & \\
\cdots \to & C_1 & \to & C_0 & \to 0 \\
 & \downarrow & & \downarrow & \\
 & 0 & & 0 &
\end{array}$$

of complexes of projective modules. (*Then,* homming into M ought to be a "correct" thing to do (no "correection terms"), and we can worry about the associated long exact sequence.)

(1/29/10) Each column would be a short exact sequence of projective modules. We know that those all split, just because the bottom guy is projective. Consequently, we may safely assume in the above that $B_i = A_i \oplus C_i$.

(1/29/10)

Lemma ("Horseshoe lemma")**.** *Let* $0 \to A \to B \to C \to 0$ *be a short exact sequence of complexes, and* $(A_i), (C_i)$ *resolutions of* A, C*, with* (C_i) *projective. Then there exists a projective resolution* $(A_i \oplus C_i)$ *of* B *and a short exact sequence of complexes (with the usual vertical maps), as pictured above.*

(2/1/10)

Proof. At each stage we have a diagram

$$\begin{array}{ccccc}
 & 0 & & 0 & \\
 & \downarrow & & \downarrow & \\
\cdots \to & A' & \stackrel{\alpha}{\to} & A'' & \to 0 \\
 & \downarrow & & \downarrow\delta & \\
\cdots \to & A' \oplus C' & \Xi\,? & B'' & \to 0 \\
 & \downarrow & & \downarrow\epsilon & \\
\cdots \to & C' & \stackrel{\gamma}{\to} & C'' & \to 0 \\
 & \downarrow & & \downarrow & \\
 & 0 & & 0 &
\end{array}$$

and we need to define a map $\Xi : A' \oplus C' \twoheadrightarrow B''$ making the squares commute. The upper square tells us it must take $(a, 0) \mapsto \delta(\alpha(a))$, so we only need to define it on $0 \oplus C'$. The lower square tells us to choose a map $\beta : C' \to B''$ with $\epsilon \circ \beta = \gamma$, and such a lift exists because C' is projective.

Now we need to be sure that the map Ξ so defined, $(a, c) \stackrel{\Xi}{\mapsto} \delta(\alpha(a)) + \beta(c)$, is surjective. Let $b \in B''$, with $\epsilon(b) \in C''$. Pick $c \in C'$ with $\gamma(c) = \epsilon(b)$. Then $(0, c) \stackrel{\Xi}{\mapsto} \beta(c) \stackrel{\epsilon}{\mapsto} \gamma(c)$, so

$\epsilon(\Xi(0,c)-b)=0$, hence $\exists a\in A'$ with $\delta(\alpha(a))=\Xi(0,c)-b$. Using these chosen a,c, we have $\Xi(-a,c)=-\delta(\alpha(a))+\beta(c)=b-\Xi(0,c)+\Xi(0,c)=b$. So yes, Ξ is surjective.

The kernel of the three surjections α,Ξ,γ is a complex

$$0\to\{a:\alpha(a)=0\in A''\}\to\{(a,c):\delta(\alpha(a))+\beta(c)=0\in B''\}\to\{c:\gamma(c)=0\in C''\}\to 0$$

and for the induction to work, we need this sequence too to be exact. The first map is $1:1$, being a restriction of the inclusion $A'\to A'\oplus C'$. The third map is onto: if $\gamma(c)=0$, then $\epsilon(\Xi(0,c))=\epsilon(\beta(c))=\gamma(c)=0$, so $\exists a\in A'$ with $\delta(\alpha(a))=\Xi(0,c)$, and $(-a,c)\in\ker\Xi$.

So it remains to check exactness in the middle. If $(a,c)\in\ker\Xi$, and $(a,c)\mapsto c=0$, then $\delta(\alpha(a))=0\in B''$. But since δ is $1:1$, $\alpha(a)=0\in A''$.

Now, when we replace the two columns in the diagram with the new complex just constructed and the zero complex, we're back in the original situation, and can use induction. □

Strictly speaking, we defined "projective" so that homming it into an exact complex was exact. Now we're trying to hom out of an exact complex of projectives.

(1/29/10) **Exercise** 2.13. Show that when we replace every A_i etc. with $\mathrm{Hom}(A_i,M)$, the result is still a short exact sequence of complexes.

(1/29/10)

Theorem 3. *There is a long exact sequence*

$$\begin{aligned}0 &\to \mathrm{Hom}(A,M)\to\mathrm{Hom}(B,M)\to\mathrm{Hom}(C,M)\\ &\to \mathrm{Ext}_1(A,M)\to\mathrm{Ext}_1(B,M)\to\mathrm{Ext}_1(C,M)\\ &\to \mathrm{Ext}_2(A,M)\to\mathrm{Ext}_2(B,M)\to\mathrm{Ext}_2(C,M)\to\cdots\end{aligned}$$

Proof. Resolve A,C using exercise 2.6. Extend to a short exact sequence of complexes using the horseshoe lemma. Construct a long exact sequence on cohomology using proposition 2. □

3.1. K^0 **of a ring.** (1/29/10) Define $K^0(R)$ as we defined $K_0(R)$, but only using projective modules. Then modding out by exact sequences amounts to imposing $[A\oplus B]=[A]+[B]$.

Exercise 3.1. Show that A,B projective implies $A\otimes_R B$ projective (again an R-module, since R is commutative).

Exercise 3.2. Show that $[A][B]:=[A\otimes_R B]$ is a well-defined product on $K^0(R)$.

Exercise 3.3. Show that it's not well-defined on $K_0(R)$.

Exercise 3.4. Show that one can use it to make $K_0(R)$ a module over $K^0(R)$.

4. Derived functors

(2/3/10) **Definition.** An **additive category** is one whose Hom-sets are abelian groups and satisfy the obvious conditions. An **additive functor** has additive maps on Hom-sets.

(2/3/10) **Example.** The category R-**Mod** of left R-modules is an additive category. Likewise the category of sheaves of $\mathcal{O}_X$-modules over a scheme X. (How does the second generalize the first? Proving your guess will require real work, and was done in [H, ch 2 somewhere].)

(2/3/10) **Definition.** A **projective object** is one satisfying the lifting property. With those, we define **projective resolutions**. Dually, we define **injective** objects and resolutions.

(2/3/10) **Definition.** If $\tau : \mathcal{C} \to \mathcal{D}$ is a contravariant functor taking exact $A \to B \to C \to 0$ to exact $0 \to \tau(C) \to \tau(B) \to \tau(A)$, we define its **right derived functors** using projective resolutions.

Why projective? Because any τ will take exact sequences that end with projectives to exact sequences. Also the horseshoe lemma gets us the short exact sequence of complexes, hence the long exact sequence on the right derived functors, so whatever we're doing is well-defined and has a chance of being computable.

So what about functors exact on the *other* side?

4.1. **Injective resolutions.** Since our goal is to study the failure of exactness of $\Gamma(X; \bullet)$, which is exact on the other side, this projective-module stuff is not exactly what we need.

(2/1/10)

Theorem 4 (we won't prove)**.** *TFAE:*

(1) $\mathrm{Hom}_R(\bullet, B)$ *is exact.*
(2) *Any injection* $B \hookrightarrow M$ *is as a direct summand.*
(3) B *is a direct summand of a "cofree" module. (***Definition.** *A* **cofree** *module is one of the form* $\mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$ *where* A *is a module.)*
(4) B *has the "lifting property" that given any* $B \leftarrow M \hookrightarrow N$*, one can extend the* $M \to B$ *map to* N *as* $M \hookrightarrow N \to B$*.*

(2/1/10)

Theorem 5 (we won't prove)**.** *Let* R *be Noetherian.*

(1) *Any module embeds in an injective module (proven by embedding it in a cofree module). One says that the category of* R*-modules has* **enough injectives***.*
(2) *Any injective module is uniquely the direct sum of indecomposable injective modules.*
(3) *The indecomposables correspond* $1:1$ *to prime ideals in* R*.*

(2/1/10) For example, in $\mathbb{Z}$, the indecomposable injectives are $\mathbb{Q}$ (for the prime ideal 0) and

$$\{m/p^n : m, n \in \mathbb{Z}\} \bmod \mathbb{Z} \quad \subseteq \quad \mathbb{Q}/\mathbb{Z}$$

for the prime ideal $p\mathbb{Z}$.

4.2. **Acyclic resolutions.** Let F be a covariant functor (contravariant works much the same, but our interest is in $\Gamma(X; \bullet)$).

(2/3/10) **Definition.** Call A F**-acyclic** if $RF_i(M) = 0$ for $i > 0$. E.g. if A is injective, we can "resolve" by $A_0 = A \to 0 \to \cdots$, apply F to $F(A) \to 0 \to \cdots$, and take cohomology to get $F(A), RF_1(A) = 0, RF_2(A) = 0, \cdots$.

(2/3/10) This may seem like a silly definition; our characterization of injective basically says that A is $\mathrm{Hom}(M, \bullet)$-acyclic for all M iff A is injective. But we might only want acyclicity for *some* M, or we might have another functor entirely, like $\Gamma(X; \bullet)$.

Theorem 6. *Let* (A_i) *be a resolution of* A *by* F*-acyclic modules. Then the cohomology of*

$$0 \to F(A_0) \to F(A_1) \to \cdots$$

is again $(RF_i(A))$.

Proof. Attach A as A_1 to make a long exact sequence $0 \to A \to A_0 \to A_1 \to \cdots$, and let $B_k \leq A_k$ be the image of $A_{k-1} \to A_k$ (with $B_0 = A$). Then for each $k \geq 0$ we have a short exact sequence

$$0 \to B_k \to A_k \to B_{k+1} \to 0 \qquad \text{with } RF_i(A_k) = 0 \text{ for } i > 0$$

giving us a long exact sequence,

$$\begin{aligned} 0 \;&\to\; F(B_k) \to F(A_k) \to F(B_{k+1}) \\ &\to\; RF_1(B_k) \to 0 \to RF_1(B_{k+1}) \\ &\to\; RF_2(B_k) \to 0 \to RF_2(B_{k+1}) \to \cdots \end{aligned}$$

with which we learn that $RF_i(B_{k+1}, M) \cong RF_{i+1}(B_k)$ for $i \geq 1, k \geq 0$. Chaining those together, we have $RF_i(A = B_0) \cong RF_1(B_{i-1})$ for $i \geq 1$.

Briefly *assume that the theorem holds for* $i = 1$. Then we can apply it to the resolution

$$0 \to B_{i-1} \to A_{i-1} \to A_i \to \cdots$$

of B_{i-1} by acyclics, to get that the 1st cohomology of this sequence is $RF_1(B_{i-1})$. But this 1st cohomology is the ith cohomology of the original sequence. So the equality above gives the result for general i.

It remains to check the original theorem specifically for $i = 1$, that $RF_1(A)$ coincides with the cohomology of

$$0 \to F(A_0) \to F(A_1) \to F(A_2) \to \cdots$$

at $F(A_1)$, namely $\ker(F(A_1) \to F(A_2))/\mathrm{img}\,(F(A_0) \to F(A_1))$. Using $0 \to B_1 \to A_1 \to A_2$ and left exactness, we get

$$0 \to F(B_1) \to F(A_1) \to F(A_2)$$

so we can rewrite that cohomology group as $F(B_1)/\mathrm{img}\,(F(A_0) \to F(A_1))$.

Whereas the first part of the sequence above, at $k = 0$, gives us

$$0 \to F(A) \to F(A_0) \to F(B_1) \to RF_1(A) \to 0$$

so $RF_1(A) \cong F(B_1)/\mathrm{img}\, F(A_0)$ also. □

5. Cohomology of sheaves

Definition. $H^i(X; \mathcal{F}) := R\Gamma_i(X; \mathcal{F})$. But this doesn't mean anything, unless:

(2/5/10)

Lemma 1. *The category of sheaves has enough injectives.*

Proof. We need to embed $\mathcal{F}$ into an injective object in the category of sheaves. For each $x \in X$, pick an injective module M_x containing the stalk $\mathcal{F}_x$, and let $\mathcal{M}_x$ be the corresponding skyscraper sheaf supported at the point x. Certainly $\mathcal{F} \to \prod_{x \in X} \mathcal{M}_x$.

Then we claim that $\mathcal{M}_x$ is injective, i.e. given $\mathcal{A} \hookrightarrow \mathcal{B}$, and $\psi : \mathcal{A} \to \mathcal{M}_x$, one can extend ψ to $\mathcal{B}$. We need to do this on each open set U, compatibly. If $U \not\ni x$, then $\Gamma(U; \mathcal{M}_x) = 0$ and the only extension is zero. If $U \ni x$, then $\Gamma(U; \mathcal{M}_x) = M_x$ and we use the fact that M_x is an injective module to extend the map from the x-stalk of $\mathcal{A}$ to one from that of $\mathcal{B}$. □

Here are some sheaves that are nicer to use:

(2/5/10) **Definition.** A sheaf $\mathcal{F}$ is **flasque** if every restriction map $\Gamma(U;\mathcal{F}) \to \Gamma(V;\mathcal{F})$ of sections on open sets $U \supseteq V$ is onto.

(2/5/10) **Non-example.** Let $U = \{(x,y) : xy = 0\}$, $V = U \setminus \vec{0}$, and $\mathcal{F}$ = the constant sheaf. Then the restriction map is not onto.

(2/8/10)

Proposition 3. [H, ex II.1.16]

(1) *If X is irreducible, then the sheaf of locally constant functions is flasque.*
(2) *The restriction of a flasque sheaf to an open set is flasque.*
(3) *If $0 \to \mathcal{F} \xrightarrow{\iota} \mathcal{A} \xrightarrow{\pi} \mathcal{B} \to 0$ is an exact sequence of sheaves, and $\mathcal{F}$ is flasque, then $\Gamma(X;\bullet)$ of this sequence is again exact.*
(4) *If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is exact, and the first two are flasque, so's the third.*

Proof. (1) Open sets are dense and connected, so the only sections are constant, and the restriction map is an isomorphism.
(2) Trivial.
(3) We already know left exactness. Let $\beta \in \Gamma(X;\mathcal{B})$. We want to show $\exists \alpha \in \Gamma(X;\mathcal{A})$ with $\pi(\alpha) = \beta$.
Consider pairs $\{(U, \alpha \in \Gamma(U;\mathcal{A}))\}$ where U is an open set over which $\pi(\alpha) = \beta$, and take a maximal one. If $U = X$, we're done. Otherwise there exists an open set $U' \not\subseteq U$ and an α' s.t. $\pi(\alpha') = \beta$ over U'. So over $U' \cap U$, $\pi(\alpha - \alpha') = 0$.
Now pick $\gamma \in \Gamma(U \cap U';\mathcal{F})$ mapping to $\alpha - \alpha'$ (which uses the left exactness of Γ), and use flasqueness to extend to $\gamma' \in \Gamma(X;U')$. Then $\alpha = \alpha' + \iota(\gamma')$ on $U \cap U'$, so α on U and $\alpha' + \iota(\gamma')$ on U' can be glued together over $U \cup U'$, mapping to β on each of U or U', contradicting maximality.
(4) Apply the previous statement to any open sets $V \subseteq U \subseteq X$, to get $\Gamma(V;\mathcal{F}) \twoheadrightarrow \Gamma(V;\mathcal{F}'')$. Then when we want to extend a section of the latter, we lift, extend, map forward.

□

(2/8/10)

Proposition 4. (1) [H, lemma III.2.4] *Injective $\mathcal{O}_X$-modules are flasque.*
(2) [H, lemma III.2.5] *Flasque $\mathcal{O}_X$-modules are acyclic.*

Proof. (1) Let $\mathcal{F}$ be an injective $\mathcal{O}_X$-module, and for any open U, let $\mathcal{O}_U$ be the extension of $\mathcal{O}_X$ by 0 off U.[3] So $0 \to \mathcal{O}_V \to \mathcal{O}_U$ is exact, and $\mathcal{F}$ is injective, hence $\mathrm{Hom}(\mathcal{O}_U, \mathcal{F}) \to \mathrm{Hom}(\mathcal{O}_V, \mathcal{F}) \to 0$ is exact. Rewriting, that is $\Gamma(\mathcal{F};U) \to \Gamma(\mathcal{F};V) \to 0$, so $\mathcal{F}$ is flasque.
(2) Let $\mathcal{F}$ be flasque, and pick a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{A} \to \mathcal{B} \to 0$$

[3]This means, if you take sections over an open set not lying inside U, you get 0. Think of it as functions on X that are required to vanish *to infinite order* off of U. These sheaves are not usually quasicoherent.

where $\mathcal{A}$ is injective hence flasque by the above, so $\mathcal{B}$ is flasque by part of the previous proposition. Then in the long exact sequence

$$\begin{array}{ccccccccc} 0 & \to & \Gamma(X;\mathcal{F}) & \to & \Gamma(X;\mathcal{A}) & \to & \Gamma(X;\mathcal{B}) & \to & \\ & \to & H^1(X;\mathcal{F}) & \to & H^1(X;\mathcal{A}) & \to & H^1(X;\mathcal{B}) & \to & \cdots \end{array}$$

we know $H^1(X;\mathcal{A}) = 0$ because $\mathcal{A}$ is injective, and the first boundary map is 0 by part of the previous proposition. Hence $H^1(X;\mathcal{F}) = 0$, but also, $H^{i+1}(X;\mathcal{F}) \cong H^i(X;\mathcal{B})$ for $i \geq 1$. Applying the theorem to $\mathcal{B}$ as well, and using induction, we get the rest. □

6. Cohomology of a Noetherian Affine Scheme

(2/10/10) It is surprisingly technical to prove the following, and we won't bother:

Proposition 5. [H, prop III.3.4] *Let* I *be an injective module over a Noetherian ring* A*. Then the sheaf* $\tilde{I}$ *over* $X := \operatorname{Spec} A$ *is flasque.*

What we already know is

- The functor $M \mapsto \widetilde{M}$ from the category of R-modules to the category of quasicoherent $\mathcal{O}_X$-modules is an equivalence, so $\tilde{I}$ is an injective object *in the category of quasicoherent* $\mathcal{O}_X$*-modules*.
- Injective objects in the category of *all* sheaves of abelian groups are flasque. (Note that this proof made use of the nonquasicoherent sheaves $\mathcal{O}_U, \mathcal{O}_V$.)
- If $\mathcal{F}$ is injective in that bigger category, it's injective in the subcategory of quasicoherent sheaves.

Unfortunately the converse of the third seems to be false for general R, though I am told it is proven for locally Noetherian R in [H2, theorem 7.18].

(2/10/10) Granting that, it's not so hard to prove

Theorem 7. [H, theorem III.3.5] *If* X *is affine,* $H^{i>0}(X;\mathcal{F}) = 0$ *for any* $\mathcal{O}_X$*-module* $\mathcal{F}$*.*

Proof. Let $M = \Gamma(X;\mathcal{F})$, so $\mathcal{F} \cong \tilde{M}$. Pick an injective resolution for M, and tilde everything to get a resolution for $\mathcal{F}$; by the proposition the objects are Γ-acyclic. So we can compute H^i using Γ of that complex, but this is just the exact sequence resolving M. □

(2/10/10) Note that this is very much at odds with the topologists' application of H^i, to the constant sheaf on X with values in $\mathbb{Z}$; affine varieties can have lots of cohomology. One should see it as a sort of Poincaré lemma for sheaf cohomology of $\mathcal{O}_X$-modules.

7. Čech cohomology of sheaves

(2/10/10) Let $\mathfrak{U} = (U_i)_{i \in I}$ be a list of open sets covering X, indexed by a well-ordered (hopefully, finite) set I. Define

$$C^p(\mathfrak{U}, \mathcal{F}) := \prod_{i_0 < \ldots < i_p} \Gamma\left(\bigcap_{j=0}^{p} U_j;\ \mathcal{F}\right)$$

and make it into a complex in the usual way:

$$(d\alpha)_{i_0,\dots,i_{p+1}} = \sum_k (-1)^k \alpha_{\text{omit } i_k}$$

The signs make everything cancel so $d^2 = 0$. Define the **Čech cohomology groups** $\check{H}(\mathfrak{U};\mathcal{F})$ from this complex.

Example 1. Let $I = \{1\}$, $U_1 = X$. Then we only have C^0, so only $\check{H}^0$, which is $\Gamma(X;\mathcal{F})$. Pretty silly, but at least that is $H^0(X;\mathcal{F})$.

(2/10/10)

Lemma 2. *Indeed,* $\check{H}^0(\mathfrak{U};\mathcal{F}) \cong H^0(X;\mathcal{F})$.

Proof. $\ker : C^0 \to C^1$ assigns a section to each U_i, in such a way that they glue together. □

(2/10/10) Since we want to compute the higher H^i, allowing this $\mathfrak{U}$ is obviously a silly state of affairs. The usual way to fix this in topology is to demand that all the U_i, and their intersections, are topologically trivial (specifically, contractible). We can't really do that in algebraic geometry, but we'll do something like that in a minute.

(2/10/10) First though, we'll "sheafify" the Čech complex, in hopes[4] that the sheaf (rather than just its global sections) remembers enough to compute correctly. Let

$$\mathcal{C}^p(\mathfrak{U};\mathcal{F}) := \prod_{i_0<\dots<i_p} \iota_*(\mathcal{F}|_{\bigcap_{j=0}^p U_j})$$

where ι is the inclusion of the open set, and ι_* is the extension by zero. Basically, any open set $V \subseteq X$ acquires its own open cover $\mathfrak{U} \cap V$, hence its own Čech complex, and they sheafify together (complete with differential).

(2/12/10)

Lemma 3. *The complex* $\mathcal{C}^p(\mathfrak{U};\mathcal{F})$ *is naturally a resolution of* $\mathcal{F}$*, i.e. there's an exact sequence*

$$0 \to \mathcal{F} \to \mathcal{C}^0 \to \mathcal{C}^1 \to \cdots$$

Proof. The first map takes a section to its restrictions on the (U_i). We can check exactness on stalks, and the nicest way to do that is with a homotopy operator $k : \mathcal{C}^p \to \mathcal{C}^{p-1}$, showing that the identity map is homotopic to the zero map.

For each point x and open set $V \ni X$, refine V to $V \cap U_j$ for some $U_j \ni x$. So now $V \subseteq U_j$. Take

$$(k\alpha)_{i_0,\dots,i_{p-1}} := \alpha_{j,i_0,\dots,i_{p-1}}$$

(introducing a -1 if we have to sort indices, and 0 if we get a repeat) and check that it satisfies $(dk + kd)(\alpha) = \alpha$.

⋆**why is that map independent of** j**?**⋆ □

(2/12/10)

Corollary 1. *If* $\mathcal{F}$ *is* Γ*-acylic for each intersection* $\bigcap_j U_j$*, then* $\check{H}^i(\mathfrak{U};\mathcal{F}) \cong H^i(X;\mathcal{F})$.

[4]"Sheafification" also goes by the name "categorification". It's a very un-well-defined process, which is part of the point; a sheaf should have lots more information than its global sections. A corresponding concept in combinatorics is to prove two numbers are equal by finding a bijective proof.

Proof. The complex above is then a resolution by Γ-acyclics, so we can use it to compute the same derived functors. Applying Γ, we recover the original complex (C^p). ☐

The replacement for "topologically trivial" we will use is "affine". One thing cool about that is this lemma:

(2/12/10)

Lemma 4. [H, ex. II.4.3] *If* X *is separated, and the* (U_i) *are affine, then so too are the intersections* $\bigcap_j U_j$.

Proof. Of course it's enough to check for two affine opens U, V. Look at the open set $U \times V \subseteq X \times X$, intersected with the diagonal $X \subseteq X \times X$. "Separated" means the diagonal is closed, so it's a closed subset of the affine variety $U \times V$, hence itself affine. ☐

This is especial motivation for showing that $\mathcal{O}_X$-modules are Γ-acyclic over affine schemes; we can use arbitrary affine covers to compute sheaf cohomology of $\mathcal{O}_X$-modules.

(2/12/10)

Corollary 2. *(Grothendieck's theorem, version 1.)*

Let X *(separated) have a cover* $\mathfrak{U}$ *by* n *affine open sets, and let* $\mathcal{F}$ *be an* $\mathcal{O}_X$*-module. Then* $H^i(X; \mathcal{F}) = 0$ *for* $i > n$.

Proof. The Čech complex resolution only goes out that far, and is acyclic. ☐

8. The Cohomology of Projective Space

Throughout this section, $\mathbb{P}^n$ denotes $\operatorname{Proj} A[x_0, \ldots, x_n]$, the n-dimensional projective space over a ring A, and $\mathcal{O}(k)$ denotes the kth tensor power of Serre's twisting sheaf.

Proposition 6. *Let* $X \subseteq \mathbb{P}^n$*, and* $\mathcal{F}$ *a* $\mathcal{O}_X$*-module. Then* $H^i(X; \mathcal{F}) = 0$ *for* $i > n + 1$.

Proof. Intersect X with $\mathbb{P}^n$'s open cover by $n + 1$ sets, and use corollary 2. ☐

(2/15/10)

Proposition 7. *Let* $X = \mathbb{P}^n$*. Then for* $k \geq 0$,

$$H^0(X; \mathcal{O}(k)) \cong \mathrm{Sym}^k(A^{n+1}), \qquad H^{i>0}(X; \mathcal{O}(k)) = 0$$

whereas for $k < 0$,

$$H^{i \neq n}(X; \mathcal{O}(k)) = 0, \qquad H^n(X; \mathcal{O}(k)) \cong \mathrm{Sym}^{-k-n-1}(A^{n+1})$$

(For $-n < k < 0$*, there is no cohomology at all.)*

Proof. Let U_i be the open set on which the projective coordinate x_i is invertible, so $\Gamma(\cap_{i \in S} U_i; \mathcal{O}(k))$ can be identified with $A[x_1, \ldots, x_n, \{x_i^{-1}\}_{i \in S}]_k$ (the subscript indicating degree k). Here $S \subseteq \{0, \ldots, n\}$.

Then Γ(the Čech complex), augmented with $A[x_1, \ldots, x_n]_k$ at the beginning, looks like

$$0 \to A[x_1, \ldots, x_n]_k \to \cdots \to \prod_{|S|=p+1} A\left[x_1, \ldots, x_n, \{x_i^{-1}\}_{i \in S}\right]_k \to \cdots \to A[x_1^{\pm}, \ldots, x_n^{\pm}]_k \to 0$$

All these modules are multigraded, so it's enough to look at one multigraded component at a time, meaning, one Laurent monomial $\prod_i x_i^{k_i}$, $\sum k_i = k$ at a time. Let $N = \{i : k_i < 0\}$. Then that subcomplex is

$$0 \to \cdots \to \prod_{|S|=p+1; S \supseteq N} A \to \cdots A \to 0.$$

There are two cases: $N = \{0, 1, \ldots, n\}$ or $\exists j \notin N$.

If $N = S$, this subcomplex (and its cohomology) is A in degree n and 0 otherwise. That contributes $\#\{(k_0, \ldots, k_n) : \sum k_i = k, \forall k_i < 0\} = \#\{(k'_0, \ldots, k'_n) : \sum k'_i = -k-n-1, \forall k'_i \geq 0\} = \binom{-k}{n}$ many copies of A to H^n.

If $j \notin N$, this complex is exact, which we leave as an exercise. (More specifically, it's the $(n - |N|)$-fold tensor power of the complex $0 \to A \to A \to 0$.)

Consequently the cohomology of this augmented complex is $Sym^{-k-n-1}(A^{n+1})$ in degree n, and otherwise 0. Chopping off the initial $A[x_1, \ldots, x_n]_k \cong Sym^k(A^{n+1})$, we get the claimed result. □

Theorem 8. [H, III.5.2] *Let $\mathcal{F}$ be a coherent sheaf on a projective scheme X over a Noetherian ring A. Then*

(1) *each $H^i(X; \mathcal{F})$ is a finitely generated A-module, and*
(2) *$H^{i>0}(X; \mathcal{F}(n)) = 0$ for $n >> 0$.*

Proof. Pick a flasque resolution of $\mathcal{F}$ over X, and push it into projective space. It's again a flasque resolution, so gives the same cohomology. Hence it's enough to prove for sheaves on projective space.

Next we claim that $\mathcal{F}(n)$ can be written as a quotient of some $\bigoplus \mathcal{O}(q_i + n)$, for $n >> 0$ [H, II.5.18]. (Proof sketch: take sections of $\mathcal{F}$ over its standard open sets to generate, and multiply them by high enough powers of $\{x_i\}$ to make them extend over the whole space.)

Tensoring with $\mathcal{O}(-n)$, we get $\mathcal{F}$ as a quotient of some $\mathcal{E} := \bigoplus \mathcal{O}(q_i)$, so $0 \to \mathcal{R} \to \mathcal{E} \to \mathcal{F} \to 0$. The kernel $\mathcal{R}$ is again coherent. This produces

$$\cdots \to H^i(X; \mathcal{E}) \to H^i(X; \mathcal{F}) \to H^{i+1}(X; \mathcal{R}) \to \cdots$$

The left is finitely generated. The right is 0 for large i, so we can do backwards induction on i. Therefore the middle is trapped between two finitely generated modules, and A is Noetherian, so it's f.g. too.

Twisting the sequence by $\mathcal{O}(n)$, we get

$$\cdots \to H^i(X; \mathcal{E}(n)) \to H^i(X; \mathcal{F}(n)) \to H^{i+1}(X; \mathcal{R}(n)) \to \cdots$$

and now the left is 0, and the right is 0 by the same backwards induction. □

Theorem 9. *(Grothendieck's theorem, version 2.) Let $\mathcal{F}$ be a sheaf on a quasiprojective scheme X of dimension n. Then $H^i(X; \mathcal{F}) = 0$ for $i > n$, indeed, for $i > \dim\ supp(\mathcal{F})$.*

Proof. Let H be a projective hypersurface containing no geometric components of the support of X. Then it is given by some equation $b = 0$, for b of degree B. Consider the exact sequence

$$0 \to \mathcal{F} \xrightarrow{\cdot b^N} \mathcal{F}(NB) \to \mathcal{C} \to 0$$

then $\mathrm{supp}(\mathcal{C}) \subseteq \mathrm{supp}(\mathcal{F}) \cap H$, so its dimension is 1 lower. Then in the sequence

$$\cdots \to H^{i-1}(X;\mathcal{C}) \to H^i(X;\mathcal{F}) \to H^i(X;\mathcal{F}(BN)) \to \cdots$$

the third vanishes for large N, and the first for $i-1 > \dim\ \mathrm{supp}(\mathcal{C})$ by induction. (The base case is empty support, of dimension -1.) □

In [H, III.2] there's a proof of this for general X. It's *really gross.*

(2/22/10) Recall $\mathcal{L}$ is **ample** on X if for any coherent sheaf $\mathcal{F}$, $\mathcal{F} \otimes \mathcal{L}^N$ is generated by global sections of $N >> 0$. Whereas $\mathcal{L}$ is **very ample over** Y if it comes from an immersion into some projective space over Y. So ample is absolute, very ample is relative (to Y).

(2/22/10)

Theorem 10. [H, II.7.6] *Let X be of finite type over A. Then $\mathcal{L}$ is ample iff some $\mathcal{L}^m$ is very ample over* $\mathrm{Spec}\, A$.

Proof sketch. $\Longleftarrow$: Let $\overline{X}$ be the closure of X in its projective immersion. Then $\mathcal{O}(1)$ is ample on $\overline{X}$. (This is [H, II.5.17], closely related to the proof sketched earlier of [H, II.5.18].) With a little work one makes it ample on X too.

$\Longrightarrow$: cover with finitely many open affine neighborhoods. Each is Spec of a finitely generated A-algebra. Each generator extends to a global section of some power. Taking the product of all those powers, we have enough sections to get an embedding. □

Proposition 8. *Let X be proper over* $\mathrm{Spec}\, A$, *A Noetherian.*

Let $\mathcal{L}$ be ample on X. Then for any coherent sheaf $\mathcal{F}$, $\mathcal{F} \otimes \mathcal{L}^N$ has no higher cohomology for $N >> 0$.

As shown in [H, III.5.3], this characterizes ampleness: hence we now have three definitions.

Proof. Since some $\mathcal{L}^m$ is very ample on X, X embeds into some projective space over $\mathrm{Spec}\, A$. Since X is proper over $\mathrm{Spec}\, A$, that embedding is closed; hence X is projective over $\mathrm{Spec}\, A$. Hence we can think of $\mathcal{L}$ as $\mathcal{O}(1)$ and apply [H, III.5.2] above. □

9. Sheaf cohomology on $\widetilde{\mathbb{P}^2}$

(2/22/10) Let $X = \{([a,b],[x,y,z]) \in \mathbb{P}^1 \times \mathbb{P}^2 : a/b = x/y\}$. Geometrically, it consists of points $[x,y,z]$ in the plane together with lines $[a,b]$ connecting them to the point $[0,0,1]$, which is to say, it is the blowup of $\mathbb{P}^2$ at that point. Let $\mathcal{O}(m,n)$ denote the pullback of $\mathcal{O}(m) \boxtimes \mathcal{O}(n)$ from $\mathbb{P}^1 \times \mathbb{P}^2$.

To compute $H^i(X;\mathcal{O}(m,n))$, we use the open cover

$$\begin{aligned} U_1 &:= \{([1,b],[1,b,z])\}, && \text{i.e. } x \text{ invertible} \\ U_2 &:= \{([a,1],[a,1,z])\}, && \text{i.e. } y \text{ invertible} \\ U_3 &:= \{([1,b],[x,bx,1])\}, && \text{i.e. } a, z \text{ invertible} \\ U_4 &:= \{([a,1],[ay,y,1])\}, && \text{i.e. } b, z \text{ invertible} \end{aligned}$$

and (as in the $\mathbb{P}^n$ calculation) a multigrading, with weights

$$wt(a) = wt(x) = (1,0)$$

$$wt(b) = wt(y) = (0,1)$$
$$wt(z) = (0,0)$$

This again has the fine[5] property that there is only one Laurent monomial of bidegree (m,n) in any given homogeneous component (remembering that we can eliminate a for bx/y, or b for ay/x).

Then for example, $\Gamma(U_1;\mathcal{O}(m,n))$ is isomorphic to the submodule of $A[x^{\pm},y,z,a,b]\langle ay-bx\rangle$ of degree m in a,b and n in degree x,y,z. Since x is invertible, we can eliminate b for ay/x, so identify it with $a^m A[x^{\pm},y,z]$ of degree n in x,y,z, and from there with $a^m x^n A[y/x,z/x]$. Under similar analysis, we see

$$\Gamma(U_1) \cong a^m x^n A[y/x, z/x]$$
$$\Gamma(U_2) \cong b^m y^n A[x/y, z/y]$$
$$\Gamma(U_3) \cong b^m z^n A[x/z, b/a]$$
$$\Gamma(U_4) \cong a^m z^n A[x/z, a/b]$$

(omitting the $\mathcal{O}(m,n)$ in each). We also need the intersections:

$$\Gamma(U_1 \cap U_2) \cong a^m x^n A[(y/x)^{\pm}, z/x]$$
$$\Gamma(U_1 \cap U_3) \cong a^m x^n A[y/x, (z/x)^{\pm}]$$
$$\Gamma(U_1 \cap U_4) \cong \Gamma(U_2 \cap U_3) \cong a^m x^n A[(y/x)^{\pm}, (z/x)^{\pm}]$$

...pictures to come...

10. Pushing around sheaves, especially by the Frobenius

10.1. Functors between categories of sheaves.

(3/1/10) Let $\tau : X \to Y$ be a morphism of schemes, and $\mathcal{F},\mathcal{G}$ be sheaves on X, Y. Then the easy thing to define, using

$$(\tau_* \mathcal{F})(V \subseteq Y) := \mathcal{F}(\tau^{-1}(V)),$$

is a "pushforward" functor τ_* from $\mathfrak{Ab}(X) \to \mathfrak{Ab}(Y)$, where $\mathfrak{Ab}(\bullet)$ is the category of sheaves of abelian groups.

(3/1/10) This turns out to have a left adjoint $f^{-1} : \mathfrak{Ab}(Y) \to \mathfrak{Ab}(X)$, meaning

$$\mathrm{Hom}_X(\tau^{-1}\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_Y(\mathcal{G}, \tau_* \mathcal{F})$$

which uniquely defines it up to unique isomorphism. So what is it? Well, it can be computed by the following slightly yucky rule: define the presheaf $U \mapsto \lim_{V \supseteq \tau(U)} \mathcal{G}(V)$, and take the associated sheaf.

Now assume that $\mathcal{F},\mathcal{G}$ are $\mathcal{O}_X-,\mathcal{O}_Y-$modules, and note that τ_* also defines a functor $\mathfrak{Mod}(X) \to \mathfrak{Mod}(Y)$ (the subcategories of $\mathcal{O}$-modules). Unfortunately $\tau^{-1}\mathcal{G}$ is not an $\mathcal{O}_X$-module; it only carries an action of $\tau^{-1}\mathcal{O}_Y$. So does $\mathcal{O}_X$, and we can define

$$\tau^* \mathcal{G} := \mathcal{O}_X \otimes_{\tau^{-1}\mathcal{O}_Y} \tau^{-1}\mathcal{G}$$

in order to get a left adjoint

$$\mathrm{Hom}_{\mathcal{O}_X}(\tau^* \mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, \tau_* \mathcal{F}).$$

One key result is the **projection formula** for $\mathcal{G}$ locally free:

$$\tau_*(\tau^* \mathcal{G} \otimes \mathcal{F}) \cong \mathcal{G} \otimes \tau_*(\mathcal{F}).$$

[5] pun

10.2. **Derived functors of pushforward.** Let $\pi : X \to \mathrm{Spec}\,\mathbf{k}$ be a map from X to a point, and $\mathcal{F}$ an $\mathcal{O}_X$-module. Then $\pi_*\mathcal{F}$ is a sheaf on the point, so just a $\mathbf{k}$-vector space, namely

$$\Gamma(\mathrm{Spec}\,\mathbf{k}; \pi_*\mathcal{F}) := \Gamma(X; \mathcal{F}).$$

Since we know $\Gamma(X; \bullet)$ is only left exact and should be supplemented by higher cohomology, we can and should do the same with any pushforward map (also left exact). We'll do this later.

10.3. **Affine maps.** (3/1/10) Recall a morphism $h : X \to Y$ is **affine** if preimages in X of open affine subsets in Y are affine [H, ex. II.5.17]. Any finite morphism is affine (by definition; the projection of the line with two origins to the line looks finitish but isn't "finite").

Lemma 5. [H, ex. III.4.1] *Let $h : X \to Y$ be affine (X, Y both separated), and let $\mathcal{F}$ be a quasicoherent sheaf on X. Then $H^i(X; \mathcal{F}) \cong H^i(Y; h_*\mathcal{F})$ for all i.*

Proof. Pick an affine open cover (U_i) of Y, giving an affine open cover $(V_i := h^{-1}(U_i))$ of X. Since $\mathcal{F}$ is quasicoherent, it is an $\mathcal{O}_X$-module, so $H^*(X; \mathcal{F})$ is calculable by the Čech complex

$$\cdots \to \oplus_{i_1 < \ldots < i_d} \Gamma(\cap U_i; \mathcal{F}) \to \cdots$$

Now let's calculate $H^i(Y; h_*\mathcal{F})$. By [H, II.5.8], $h_*\mathcal{F}$ is again quasicoherent, so calculable by the Čech complex

$$\cdots \to \oplus_{i_1 < \ldots < i_d} \Gamma(\cap V_i; h_*\mathcal{F}) \to \cdots$$

but these are the same groups. □

Let $\mathbf{k}$ be a **perfect** field of characteristic p, meaning the Frobenius endomorphism $F : x \mapsto x^p$ is surjective.

Lemma 6. [BrKu05, 1.1.1] (3/5/10) *Let A be a localization of a finitely generated $\mathbf{k}$-algebra, and A^p the image of the Frobenius. Then A is finitely generated over A^p.*

Proof. If $A = \mathbf{k}[x_1, \ldots, x_n]$, then the p^n monomials with no exponent $\geq n$ are a basis. For quotients of that, they're a generating set. If S is a multiplicative subset of A, then $S^{-1}A = (S^p)^{-1}A$ and $(S^{-1}A)^p = (S^p)^{-1}A^p$, so $S^{-1}A$ is f.g. over $(S^{-1}A)^p$. □

Non-example. $A = \mathbf{k}[x_1, x_2, \ldots]$.

Now let X be a scheme defined over $\mathbb{F}_p$. Let $F : X \to X$ denote the **absolute Frobenius**, acting as the identity on the space X and the pth power map on $\mathcal{O}_X$.

Corollary 3. *Let X be quasiprojective over $\mathbb{F}$. Then F is finite.*

Lemma 7. [BrKu05, 1.2.6] (3/8/10) *Let $\mathcal{L}$ be an invertible sheaf on X. Then*

$$F^*\mathcal{L} \cong \mathcal{L}^{\otimes p}, \qquad \textit{and} \qquad F_*(F^*\mathcal{L}) \cong \mathcal{L} \otimes_{\mathcal{O}_X} F_*\mathcal{O}_X.$$

Proof. By definition [H, II.5],

$$F^*\mathcal{L} := F^{-1}\mathcal{L} \otimes_{F^{-1}\mathcal{O}_X} \mathcal{O}_X.$$

Since F is the identity on points, the first two ingredients are just $\mathcal{L}$ and $\mathcal{O}_X$ again; the only interesting part is the action map $F^{-1}\mathcal{O}_X \to \mathcal{O}_X$ coming from $F^\# : \mathcal{O}_X \to F_*\mathcal{O}_X$. So

$$F^*\mathcal{L} = \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_X$$

where $\sigma f \otimes g = \sigma \otimes f^p g$ for local sections σ of $\mathcal{L}$ and f, g of $\mathcal{O}_X$. Thus the map

$$F^*\mathcal{L} \to \mathcal{L}^p, \quad \sigma \otimes f \mapsto \sigma^p f$$

is well-defined and $\mathcal{O}_X$-linear. Being a surjective $\mathcal{O}_X$-linear map of invertible sheaves, it is an isomorphism.

For the second result,

$$F_*(F^*\mathcal{L}) \cong \mathcal{L} \otimes_{\mathcal{O}_X} F_*(F^*\mathcal{O}_X) \cong \mathcal{L} \otimes_{\mathcal{O}_X} F_*(\mathcal{O}_X).$$

□

10.4. **Separated and proper maps.** (3/1/10) A map $f : X \to Y$ is **separated** if the diagonal inclusion into $X \times_Y X$ is closed. A scheme X is separated (no map) if this is true for the map to $\operatorname{Spec} \mathbb{Z}$.

(3/1/10) A map $f : X \to Y$ is **proper** if it is separated, of finite type, and **universally closed**: not only should it take closed subsets to closed subsets, but given any base change $Y' \to Y$, the associated pullback $X' \to Y'$ should also take closed subsets to closed subsets.

What does this have to do with the definition of "proper" in topology? For Y a point, topology says X should be compact. If X is a noncompact manifold, then we can map $(0, \infty) \to X$ with closed image, wandering off the end of X; choose a map w. Now look at the base change $X \times [0, \infty) \to [0, \infty)$, and observe that the closed subset $\{(w(t), t) : t \in (0, \infty)\}$ has non-closed image.

11. A first look at Frobenius splitting

(2/26/10) Throughout this section X is defined over a perfect field $\mathbb{F}$ of characteristic p. ("Perfect" means that the pth power map, the **Frobenius endomorphism**, is not only $1:1$ but onto, e.g. if $\mathbb{F}$ is finite or algebraically closed.)

11.1. $X = \operatorname{Spec} R$. (2/26/10) A **Frobenius splitting** of a ring R over $\mathbb{F}$ is a map $\phi : R \to R$, a sort of pth root, satisfying

(1) $\phi(a + b) = \phi(a) + \phi(b)$
(2) $\phi(a^p b) = a\phi(b)$
(3) $\phi(1) = 1$.

Without condition (3) we call it a **near-splitting**. Conditions (1) and (2) are easy to satisfy – $\phi \equiv 0$ works – so really, the hard condition is (3).

(3/1/10) If R is equipped with a splitting φ, we will say R is **split** (not just "splittable"; we care about the choice of φ). Call an ideal $I \leq R$ of a ring with a Frobenius (near-)splitting φ **compatibly (near-)split** if $\varphi(I) \subseteq I$. For the convenience of the reader we recapitulate the basic results of Frobenius splitting we will use:

Theorem. [BrKu05, section 1.2] *Let R be a Frobenius split ring with ideals I, J.*

(1) *R is reduced.*
(2) *If I is compatibly split, then I is radical, and $\varphi(I) = I$.*
(3) *If I and J are compatibly split ideals, then so are $I \cap J$ and $I + J$. Hence they are radical.*
(4) *If I is compatibly split, and J is arbitrary, then $I : J$ is compatibly split. In particular the prime components of I are compatibly split.*

Note that the sum of radical ideals is frequently *not* radical; "compatibly split" is a much more robust notion.

Proof. (1) Assume not, and let r be a nonzero nilpotent with m chosen largest such that $r^m \neq 0$ but $r^{m+1} = 0$. Let $s = r^m$. Then $0 = s^p$, so $0 = \varphi(s^p) = s$, contradiction.
(2) If I is compatibly split, then φ descends to a splitting of R/I, so R/I is reduced. Equivalently, I is radical. Since I contains $\{i^p : i \in I\}$, one always has $\varphi(I) \supseteq I$.
(3) $\varphi(I \cap J) \subseteq \varphi(I) \cap \varphi(J) \subseteq I \cap J$. $\varphi(I + J) \subseteq \varphi(I) + \varphi(J)$ because φ is additive.
(4) $r \in I : J \iff \forall j \in J, rj \in I \implies \forall j \in J, rj^p \in I \implies \forall j \in J, \varphi(rj^p) \in I$ (since I is compatibly split) $\iff \forall j \in J, \varphi(r)j \in I \iff \varphi(r) \in I : J$.

□

Corollary 4. *Let I be a compatibly split ideal in a Frobenius split ring. From it we can construct many more ideals, by taking prime components, sums, and intersections, then iterating. All of these will be radical.*

It was recently observed [Schw, KuMe], and only a little harder to prove (a few pages, rather than a few lines), that a Noetherian split ring R has only finitely many compatibly split ideals. In very special cases the algorithm suggested in corollary 4 finds all of them.

(2/26/10) Let $\mathbb{F}$ be a perfect field over $\mathbb{F}_p$. There is a near-splitting on $\mathbb{F}[x_1, \ldots, x_n]$ called $\mathrm{Tr}(\bullet)$ uniquely characterized by its application to monomials m:

$$\mathrm{Tr}(m) = \begin{cases} \sqrt[p]{m \prod_i x_i} \Big/ \prod_i x_i & \text{if } m \prod_i x_i \text{ is a } p\text{th power} \\ 0 & \text{otherwise.} \end{cases}$$

The **standard splitting** of $\mathbb{F}[x_1, \ldots, x_n]$ is $\varphi(g) := \mathrm{Tr}\left((\prod_{i=1}^n x_i)^{p-1} g\right)$. It "takes the pth root where possible".

Lemma 8. (1) *The standard splitting is a Frobenius splitting, and the ideals that it compatibly splits are exactly the* **Stanley-Reisner ideals** *(meaning, those generated by squarefree monomials).*
(2) (2/26/10) *Every near-splitting on $\mathbb{F}[x_1, \ldots, x_n]$ is of the form $\mathrm{Tr}(g\bullet)$ for some $g \in \mathbb{F}[x_1, \ldots, x_n]$. If $g = f^{p-1}$, then $\langle f \rangle$ is near-split by $\mathrm{Tr}(g\bullet)$.*

If $R[S^{-1}]$ is a localization of a split ring R, it too is split, by the unique rule $\phi(r/s) := \phi(rs^{p-1}/s^p) = \phi(rs^{p-1})/s$ under which the map $R \to R[S^{-1}]$ intertwines the two ϕ. This localization means that the concept sheafifies.

11.2. **First examples.** (3/5/10) Let $P \subseteq \mathbb{R}^n$ be a convex polytope with rational vertices. Let $CP = \overline{\mathbb{R}_+ \cdot (P \times \{1\})}$ be the cone on P, and $\mathbf{k}[CP \cap \mathbb{Z}^{n+1}]$ be the monoid algebra of the lattice points in this cone. Then the **projective toric variety** X_P is defined to be Proj of this ring, where the grading comes from the last coordinate.

(Not every "toric variety" arises in this way; not even every complete one.)

Example 2. (1) If P is the standard n-simplex, X_P is $\mathbb{P}^n_{\mathbf{k}}$.
(2) If P is a product $Q \times R$, $X_P \cong X_Q \times X_R$.
(3) If P is the trapezoid with vertices $(1,0), (0,1), (2,0), (0,2)$, X_P is the blowup of $\mathbb{P}^2$ at a point.

Theorem 11. *X_P is Frobenius split.*

Proof. Extend the standard splitting to $\mathbf{k}[x_1^{\pm}, \ldots, x_{n+1}^{\pm}]$, and restrict it to the coordinate ring of X_P. □

11.3. **Applications to sheaf cohomology.** (3/8/10) A **Frobenius splitting** is a map

$$\phi : F_* \mathcal{O}_X \to \mathcal{O}_X$$

such that the composite $\phi \circ F^\#$ is the identity map of $\mathcal{O}_X$. (Here $F^\# : \mathcal{O}_X \to F_X \mathcal{O}_X$ is just the pth power map.)

A **compatibly split** subscheme Y is one such that $\phi(F_* \mathcal{I}_Y) \subseteq \mathcal{I}_Y$, where $\mathcal{I}_Y$ is the ideal sheaf. All the properties of compatibly split ideals, that are local properties of schemes, are therefore also true for such subschemes (e.g., reducedness).

Theorem 12. [BrKu05, 1.2.7-8] *(3/8/10) Let X be split, and $\mathcal{L}$ an ample line bundle on X.*

(1) $H^{i>0}(X; \mathcal{L}) = 0$.
(2) *If Y is a compatibly split subscheme, then the restriction map $H^0(X; \mathcal{L}) \to H^0(Y; \mathcal{L})$ is onto.*

Proof. (1) Consider the injection of sheaves

$$\mathcal{L} \xrightarrow{\sim} \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_X \xrightarrow{1 \otimes F^\#} \mathcal{L} \otimes_{\mathcal{O}_X} F_* \mathcal{O}_X.$$

This latter injection is split (in the usual sense) by the map $1 \otimes \phi$. Hence

$$H^i(1 \otimes F^\#) : H^i(X; \mathcal{L}) \to H^i(X; \mathcal{L} \otimes_{\mathcal{O}_X} F_* \mathcal{O}_X)$$

is injective. But

$$H^i(X;\ \mathcal{L} \otimes_{\mathcal{O}_X} F_* \mathcal{O}_X) \quad \cong \quad H^i(X;\ F_*(F^* \mathcal{L})) \quad \cong \quad H^i(X;\ F_*(\mathcal{L}^{\otimes p})) \qquad \cong \qquad H^i(X;\ \mathcal{L}^{\otimes p})$$

where the last isomorphism uses that the map F is finite (by [BrKu05, 1.1.1], proved above) hence affine. (That isomorphism is only $\mathbb{F}_p$-linear.)

Hence $H^i(X;\ \mathcal{L})$ injects into $H^i(X;\ \mathcal{L}^{\otimes p})$, and iterating, into $H^i(X;\ \mathcal{L}^{\otimes p^N})$, which is eventually 0.

(2) Consider the commuting diagram (of $\mathbb{F}_p$-modules)

$$\begin{array}{ccc} H^0(X; \mathcal{L}) & \to & H^0(X; \mathcal{L}^{\otimes p^N}) \\ \downarrow & & \downarrow \\ H^0(Y; \mathcal{L}) & \to & H^0(Y; \mathcal{L}^{\otimes p^N}). \end{array}$$

By the above, the horizontal arrows are split injections. So it is enough to know that $H^0(X; \mathcal{L}^{\otimes p^N}) \twoheadrightarrow H^0(Y; \mathcal{L}^{\otimes p^N})$ for N large.

Consider the short exact sequence

$$0 \to \mathcal{I}_Y \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes p^N} \to \mathcal{L}^{\otimes p^N} \to \mathcal{L}^{\otimes p^N}|_Y \to 0$$

inducing

$$\cdots \to H^0(X; \mathcal{L}^{\otimes p^N}) \to H^0(X; \mathcal{L}^{\otimes p^N}|_Y) \to H^0(X; \mathcal{I}_Y \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes p^N}) \to \cdots;$$

the middle one is $\cong H^0(Y; \mathcal{L}^{\otimes p^N})$ and the last vanishes for large N by Serre's vanishing theorem.

□

(3/10/10) One of the ways to define the set $\mathrm{Ext}^i(A,B)$ is with exact sequences $0 \to A \to M_1 \to \ldots \to M_i \to B \to 0$, modulo chain morphisms that are the identity on A and B. It's then trickier to define the additive group structure (as we did for the $i = 1$ case before).

Why do this?

- Defining it as derived functors of Hom requires us to have enough injectives/projectives.
- By splicing complexes together, we get product maps

$$\mathrm{Ext}^i(A,B) \times \mathrm{Ext}^j(B,C) \to \mathrm{Ext}^{i+j}(A,C)$$

which could be useful.

On $\mathbf{Mod}(X)$, there are two different Hom functors we could right derive, $\mathrm{Hom}_X(\mathcal{F}, \bullet) : \mathbf{Mod}(X) \to \mathfrak{Ab}$ and $\mathcal{H}om_X(\mathcal{F}, \bullet) : \mathbf{Mod}(X) \to \mathbf{Mod}(X)$. (The first is the global sections of the second.) Call their right derived functors $\mathrm{Ext}^i(\mathcal{F}, \bullet)$ and $\mathcal{E}xt^i(\mathcal{F}, \bullet)$. (It is *not* true that the first is the global sections of the second.)

Proposition 9. [H, III.6.3] *(3/10/10)*

(1) $\mathcal{E}xt^0(\mathcal{O}_X, \mathcal{G}) = \mathcal{G}, \quad \mathcal{E}xt^{i>0}(\mathcal{O}_X, \mathcal{G}) = 0.$
(2) $\mathrm{Ext}^i(\mathcal{O}_X, \mathcal{G}) = H^i(X, \mathcal{G})$.

Proof. Since $\mathcal{H}om_X(\mathcal{O}_X, \bullet)$ is the identity functor, it is exact, which gives the first one. The functor $\mathrm{Hom}_X(\mathcal{O}_X, \bullet)$ is naturally isomorphic to $\Gamma(X; \bullet)$ so they have the same right derived functors. $\square$

In [H, ex. III.6.2] we see that $\mathbf{Mod}(X)$ doesn't have enough projectives, so we can't define these by left-deriving $\mathrm{Hom}_X(\bullet, \mathcal{G}), \mathcal{H}om_X(\bullet, \mathcal{G})$.

Factoid we'll prove in a moment: if $\mathcal{L}$ is a locally free sheaf of finite rank,

$$\mathrm{Ext}^i(\mathcal{F}, \mathcal{G}) \cong \mathrm{Ext}^i(\mathcal{F} \otimes \mathcal{L}, \mathcal{G} \otimes \mathcal{L}).$$

(3/10/10) With this, we can analyze the Ext-products on $\mathbb{P}^d_{\mathbf{k}}$:

$$\mathrm{Ext}^i(\mathcal{O}(m), \mathcal{O}(n)) \otimes \mathrm{Ext}^j(\mathcal{O}(n), \mathcal{O}(p)) \to \mathrm{Ext}^{i+j}(\mathcal{O}(m), \mathcal{O}(p))$$

$$\mathrm{Ext}^i(\mathcal{O}, \mathcal{O}(n-m)) \otimes \mathrm{Ext}^j(\mathcal{O}, \mathcal{O}(p-n)) \to \mathrm{Ext}^{i+j}(\mathcal{O}, \mathcal{O}(p-m))$$

$$H^i(\mathcal{O}(n-m)) \otimes H^j(\mathcal{O}(p-n)) \to H^{i+j}(\mathcal{O}(p-m))$$

Now there are cases. If $i, j, i+j \notin \{0, d\}$, then both sides are 0. So the interesting possibilities are $(0,0,0)$, $(0,d,d)$, and $(d,0,d)$ (which turns out to be symmetric).

$$i = j = 0 : \mathrm{Sym}^{n-m}(\mathbf{k}^{d+1}) \otimes \mathrm{Sym}^{p-n}(\mathbf{k}^{d+1}) \to \mathrm{Sym}^{p-m}(\mathbf{k}^{d+1})$$

which is only nonzero if $n - m, p - m \geq 0$. Then it can be identified with multiplication of polynomials.

$$i = 0, j = d : \mathrm{Sym}^{n-m}(\mathbf{k}^{d+1}) \otimes \mathrm{Sym}^{1-d-p+n}(\mathbf{k}^{d+1}) \to \mathrm{Sym}^{1-d-p+m}(\mathbf{k}^{d+1})$$

This can be identified with applying differential operators to polynomials, obtaining polynomials of lower degree. Special case $p = m - d + 1$:

$$\mathrm{Ext}^0(\mathcal{O}(m), \mathcal{O}(n)) \otimes \mathrm{Ext}^d\left(\mathcal{O}(n), \mathcal{O}(m) \otimes \mathcal{O}(-d+1)\right) \to \mathbf{k}$$

defines a perfect pairing between these Ext groups. This is the prototype of Serre duality.

To prove various that various things that look like they should compute cohomology actually do, [H] uses the following trick over and over in this section (but he doesn't prove it, and I'm not going to either):

Lemma 9. [H, III.1.3-4] *Let $(T^i)_{i\geq 0}$ be a covariant δ-functor, meaning it has boundary morphisms $T^i(C) \to T^{i+1}(A)$ for any short exact sequence $0 \to A \to B \to C \to 0$, inducing long exact sequences, functorially.*

If for each T^i and each object A, one can embed $A \hookrightarrow M$ with $T^i(M) = 0$, then the (T^i) are the right derived functors of T^0.

For example, we want to prove that the definition of $\mathcal{E}xt^i$ is local.

Lemma 10. [H, III.6.1-2] *For any open set U we have*

$$\mathcal{E}xt^i_X(\mathcal{F}, \mathcal{G})|_U \cong \mathcal{E}xt^i_U(\mathcal{F}|_U, \mathcal{G}|_U).$$

Proof. Both sides are δ-functors (in the $\mathcal{G}$ slot) from $\mathbf{Mod}(X) \to \mathbf{Mod}(U)$, and agree for $i = 0$ (where they are Hom). If we embed $\mathcal{G}$ into an injective sheaf $\mathcal{E}$, the LHS vanishes. If we knew $\mathcal{E}|_U$ were again injective, then the RHS would vanish too, and we could apply lemma 9.

So say $\mathcal{A} \hookrightarrow \mathcal{B}$, and $\mathcal{A} \to \mathcal{E}|_U$, in $\mathbf{Mod}(U)$. Thus $j_!\mathcal{A} \hookrightarrow j_!\mathcal{B}$ and $j_!\mathcal{A} \to j_!(\mathcal{E}|_U) \hookrightarrow \mathcal{E}$, in $\mathbf{Mod}(X)$, where $j_!$ is extension by 0. Hence the latter extends to $j_!\mathcal{B} \to \mathcal{E}$ by injectivity on X, which restricts to a map $\mathcal{B} \to \mathcal{E}|_U$. □

Corollary 5. *If $\mathcal{L}$ is locally free of finite rank,*

$$\mathrm{Ext}^i(\mathcal{F} \otimes \mathcal{L}, \mathcal{G}) \cong \mathrm{Ext}^i(\mathcal{F}, \mathcal{G} \otimes \mathcal{L}^\vee)$$

and

$$\mathcal{E}xt^i(\mathcal{F} \otimes \mathcal{L}, \mathcal{G}) \cong \mathcal{E}xt^i(\mathcal{F}, \mathcal{G} \otimes \mathcal{L}^\vee) \cong \mathcal{E}xt^i(\mathcal{F}, \mathcal{G}) \otimes \mathcal{L}^\vee.$$

We figured out how to compute H^i via acyclic resolutions. How can we extend that to $\mathcal{E}xt^i$ sheaves?

Proposition 10. *Given a resolution $\cdots \to \mathcal{L}_1 \to \mathcal{L}_0 \to \mathcal{F} \to 0$ by locally free sheaves, we can compute $\mathcal{E}xt^i$ as the ith cohomology of $\mathcal{H}om(\mathcal{L}_\bullet, \mathcal{G})$.*

Proof. Both sides are δ-functors in $\mathcal{G}$, agree at $i = 0$, vanish for $i > 0$ and $\mathcal{G}$ injective, so by lemma 9 agree. □

How are $\mathcal{E}xt$ and Ext connected?

Proposition 11. [H, III.6.8] *Let $\mathcal{F}$ be coherent, and $\mathcal{G}$ any $\mathcal{O}_X$-module, on X Noetherian. Then at each point x the stalk of $\mathcal{E}xt^i$ may be computed as*

$$\mathcal{E}xt^i(\mathcal{F}, \mathcal{G})_x \cong \mathrm{Ext}^i_{\mathcal{O}_x}(\mathcal{F}_x, \mathcal{G}_x)$$

where the RHS is over the local ring $\mathcal{O}_x$.

Proof. Since $\mathcal{E}xt^i$ can be computed locally, we can assume X affine. Then $\mathcal{F}$ has a free resolution, which gives free resolutions at stalks. Use proposition 10 to compute the LHS.

The RHS is actually a sheaf $\mathcal{E}xt$, over $\mathrm{Spec}\,\mathcal{O}_x$. The stalk functor is exact, and takes the free resolution used for the LHS to a free resolution used for the RHS. □

Proposition 12. [H, III.6.9] *Let X be projective over A Noetherian, and $\mathcal{F}, \mathcal{G}$ coherent. Then for large n,*

$$\mathrm{Ext}^i(\mathcal{F}, \mathcal{G}(n)) \cong \Gamma\left(\mathcal{E}xt^i(\mathcal{F}, \mathcal{G}(n))\right).$$

Proof. *Case I:* $i = 0$. Then this is about Homs.

Case II: $i > 0, \mathcal{F} = \mathcal{O}_X$. The LHS is $H^i(\mathcal{G}(n))$ so 0 for large n. Whereas the RHS is always 0.

Case III: $i > 0, \mathcal{F}$ locally free. Then we can pull it into the second factor as $\mathcal{F}^\vee$.

Case IV: $i > 0, \mathcal{F}$ general. Write it as a quotient of a locally free sheaf, $0 \to \mathcal{R} \to \mathcal{E} \to \mathcal{F} \to 0$. Then we have an exact sequence

$$0 \to \mathrm{Hom}(\mathcal{F}, \mathcal{G}(n)) \to \mathrm{Hom}(\mathcal{E}, \mathcal{G}(n)) \to \mathrm{Hom}(\mathcal{R}, \mathcal{G}(n)) \to \mathrm{Ext}^1(\mathcal{F}, \mathcal{G}(n)) \to 0$$

and isomorphisms $\mathrm{Ext}^i(\mathcal{R}, \mathcal{G}(n)) \to \mathrm{Ext}^{i+1}(\mathcal{F}, \mathcal{G}(n))$, similarly for $\mathcal{H}om, \mathcal{E}xt^i$.

If we twist the sheaf sequence by $\mathcal{O}(\text{large})$, Γ of it becomes exact, and $\mathcal{R}$ is also coherent, ... This is like the argument that $H^i(\mathcal{F}(n)) = 0$ for n large. □

13. The Serre Duality Theorem

Let $f : X \to Y$ be a morphism, $\Delta : X \to X \times_Y X$ the diagonal, $\mathcal{I}$ be the sheaf on $X \times_Y X$ of functions vanishing on the diagonal $\Delta(X)$, and $\Omega_{X/Y} := \Delta^*(\mathcal{I}/\mathcal{I}^2)$ the **sheaf of relative differentials**.

Most familiar case: Y is a point $\mathrm{Spec}\,\mathbf{k}$ and X is a vector space. Then $\mathcal{I}$ is functions on X^2 vanishing on the diagonal, and can pointwise on $\Delta(X)$ be thought of as function on some complementary subspace to $\Delta(X)$, isomorphic to X, vanishing at the origin. Then $\mathcal{I}/\mathcal{I}^2$ says we only care about the linear part of those functions. So at each point on X, we get a copy of X^*, and $\Omega_{X/Y}$ is the cotangent bundle.

That same analysis works formally at each regular point of X, so if X is regular, this sheaf is just the cotangent bundle. The **canonical sheaf** $\omega = \mathrm{Alt}^n \Omega_{X/k}$ is used also in differential topology, where it is called "the line bundle of volume forms", and one fixes a nonvanishing section of it to define an orientation and study Poincaré duality.

Theorem 13. [H, III.7.1] *Let $X = \mathbb{P}^n_{\mathbf{k}}$ over a field $\mathbf{k}$, and $\omega = \mathrm{Alt}^n \Omega_{X/k}$ be the canonical sheaf.*

(1) $\omega \cong \mathcal{O}(-n-1)$. *Hence $H^n(X; \omega_X) \cong \mathbf{k}$. Fix an isomorphism (this is an analogue of picking a volume form).*
(2) *For any coherent sheaf $\mathcal{F}$ on X, there is a natural pairing*

$$\mathrm{Hom}(\mathcal{F}, \omega) \times H^n(X, \mathcal{F}) \to H^n(X, \omega) \cong \mathbf{k}$$

and it is perfect.
(3) *For every $i \geq 0$ there is a functorial isomorphism*

$$\mathrm{Ext}^i(\mathcal{F}, \omega) \cong H^{n-i}(X, \mathcal{F})^*$$

which for $i = 0$ is the the duality above.

Proof. (1) The cotangent bundle fits into a short exact sequence

$$0 \to \Omega_{X/\mathrm{Spec}\,\mathbf{k}} \to \oplus^{n+1} \mathcal{O}(-1) \to \mathcal{O} \to 0$$

explained lovingly in coordinates in [H, II.8.13]. Rather than recapitulate that, think about the tangent space to $\ell \in X$, by pulling it back to $\mathbb{A}^{n+1} \setminus \vec{0}$, where the tangent space looks like $\mathbb{A}^{n+1}/\ell$. Thinking about that hard enough, we get the dual sequence to the above.

Once we do that, we use our known calculation of sheaf cohomologies on $\mathbb{P}^n$.

(2) Why is there a map? If we e.g. use Čech cohomology, we can take a sequence that computes $H^n(X, \mathcal{F})$, apply the morphism $\mathcal{F} \to \omega$ on the individual groups, and get a chain morphism that descends to a map $H^n(X, \mathcal{F}) \to H^n(X, \omega)$.

If $\mathcal{F} = \mathcal{O}(q)$, we've done this calculation (or can at least appeal to representation theory). Then the same works for $\mathcal{E}_0 = \bigoplus^{\text{finite}} \mathcal{O}(q_i)$. For $\mathcal{F}$ a quotient of $\mathcal{E}_0$ by the image of some $\mathcal{E}_1$, we can apply the left-exact functors $\mathrm{Hom}(\bullet, \omega)$ and $H^n(X, \bullet)^*$ to

$$\mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{F} \to 0$$

and apply the 5-lemma. `http://en.wikipedia.org/wiki/Five_lemma`

(3) Both sides are contravariant δ-functors, and we just checked the $i = 0$ part. So we need to know that both functors die for $i > 0$ when applied to the right something that quotients to $\mathcal{F}$. That turns out to be $\bigoplus \mathcal{O}(q_i << 0)$.

□

A **dualing sheaf** ω_X° on X, proper over $\mathrm{Spec}\, \mathbf{k}$, is a coherent sheaf with a **trace morphism** $t : H^{\dim X}(X, \omega_X^\circ) \to \mathbf{k}$ such that for all coherent sheaves $\mathcal{F}$,

$$\mathrm{Hom}(\mathcal{F}, \omega_X^\circ) \times H^n(X, \mathcal{F}) \to H^n(X, \omega_X^\circ) \xrightarrow{t} \mathbf{k}$$

induces an isomorphism

$$\mathrm{Hom}(\mathcal{F}, \omega_X^\circ) \to H^n(X, \mathcal{F})'$$

of (finite-dimensional) vector spaces over $\mathbf{k}$. The theorem above says that ω_X is a dualizing sheaf for $X = \mathbb{P}^n$.

Theorem 14. [H, III.8.2] *If* X *is proper, then dualizing sheaves on* X *are unique up to unique isomorphism, in the only reasonable sense.*

Proof. The Yoneda lemma `http://en.wikipedia.org/wiki/Yoneda_lemma`. The functor being represented is the contravariant functor $\mathcal{F} \mapsto H^n(X, \mathcal{F})'$ from coherent sheaves to vector spaces over $\mathbf{k}$. □

That's uniqueness. Following [H], we'll prove existence for projective X, even though they exist for arbitrary complete X. Obviously this involves stealing the dualizing sheaf from $\mathbb{P}^n_{\mathbf{k}}$, where we're already done. As such it's natural to measure the difference between X and $P = \mathbb{P}^n_{\mathbf{k}}$ by the codimension.

Lemma 11. [H, III.7.3-4] *Let* $X \subseteq P = \mathbb{P}^N_{\mathbf{k}}$ *be a subscheme of codimension* r.

(1) $\mathcal{E}xt^i_P(\mathcal{O}_X, \omega_P) = 0$ *for* $i < r$.
(2) *Let* $\omega_X^\circ = \mathcal{E}xt^r_P(\mathcal{O}_X, \omega_P)$. *Then for any* $\mathcal{O}_X$*-module* $\mathcal{F}$,

$$\mathrm{Hom}_X(\mathcal{F}, \omega_X^\circ) \cong \mathrm{Ext}^r_P(\mathcal{F}, \omega_P).$$

Proof. (1) Let $\mathcal{F}^i = \mathcal{E}xt^i_P(\mathcal{O}_X, \omega_P)$, necessarily coherent (being an $\mathcal{E}xt$ of coherent sheaves on a Noetherian scheme; use proposition 10 to compute $\mathcal{E}xt^i$). Tensoring with $\mathcal{O}(q >> 0)$

(an invertible operation), we get a sheaf generated by global sections. So to show the sheaf vanishes, it is enough to show $\Gamma(P, \mathcal{F}^i(q)) = 0$ for $q >> 0$. We know

$$\begin{aligned}
\Gamma(P, \mathcal{F}^i(q)) &= \Gamma(P, \mathcal{E}xt^i_P(\mathcal{O}_X, \omega_P)(q)) && \text{by proposition 12} \\
&\cong \Gamma(P, \mathcal{E}xt^i_P(\mathcal{O}_X, \omega_P(q))) \\
&\cong \mathrm{Ext}^i_P(\mathcal{O}_X, \omega_P(q)) \\
&\cong H^{N-i}(P, \mathcal{O}_X(-q))' && \text{duality for } \mathbb{P}^N
\end{aligned}$$

The support of this sheaf is $N - r$-dimensional, so if $N - i > N - r$, this cohomology vanishes by our wimpy version of Grothendieck's theorem.

(2) *Warning:* this is the most sketchy proof that I remember finding in [H]. Hold on tight.

Note: if $\tau : \mathcal{F} \to \mathcal{I}$ is a morphism of $\mathcal{O}_P$-modules, it factors through the $\mathcal{O}_X$-module $\mathcal{J} := \mathcal{H}om_P(\mathcal{O}_X, \mathcal{I})$, by $\sigma \mapsto (f \mapsto f\tau(\sigma)) \mapsto \tau(\sigma)$. This sets up an isomorphism between $\mathrm{Hom}_P(\mathcal{F}, \mathcal{I})$ and $\mathrm{Hom}_X(\mathcal{F}, \mathcal{J})$.

Pick an injective resolution $0 \to \omega_P \to \mathcal{I}^\bullet$ and use it to compute $\mathrm{Ext}^i_P(\mathcal{F}, \omega_P)$ as the cohomology of the complex $\mathrm{Hom}_P(\mathcal{F}, \mathcal{I}^\bullet)$. Since $\mathcal{F}$ is an $\mathcal{O}_X$-module, any element of $\mathrm{Hom}_P(\mathcal{F}, \mathcal{I}^i)$ factors as $\mathcal{F} \to \mathcal{H}om_P(\mathcal{O}_X, \mathcal{I}^i) \to \mathcal{I}^i$. Let $\mathcal{J}^i$ be this middle guy.

We got from $0 \to \omega_P \to \mathcal{I}^\bullet$ to $0 \to \mathcal{H}om_P(\mathcal{O}_X, \omega_P) \to \mathcal{J}^\bullet$, so we can compute $\mathcal{E}xt^r_P(\mathcal{O}_X, \omega_P)$ from the rth cohomology of the complex of $\mathcal{J}^\bullet$. By part (1), h^i of $\mathcal{J}^i$ is zero for $i < r$, so the complex is exact there. But the $\mathcal{J}^i$ are again injective (as $\mathcal{O}_X$-modules). (Proof: $\mathrm{Hom}_X(\mathcal{F}, \mathcal{J}^i) \cong \mathrm{Hom}_P(\mathcal{F}, \mathcal{I}^i)$, and $\mathrm{Hom}_P(\bullet, \mathcal{I}^i)$ is exact, so $\mathrm{Hom}_X(\bullet, \mathcal{J}^i)$ is too.) This buys us that the complex is *split* exact for $i < r$. Hence $\mathcal{J}^\bullet \cong \mathcal{J}_1^\bullet \oplus \mathcal{J}_2^\bullet$, where $\mathcal{J}_1$ is supported in degrees $\leq r$ and is exact, and $\mathcal{J}_2$ is supported in degrees $\geq r$.

Therefore its rth cohomology is $\ker(\mathcal{J}_2^r \to \mathcal{J}_2^{r+1})$. Hence $\omega_X^\circ = \ker(\mathcal{J}_2^r \to \mathcal{J}_2^{r+1})$.

By the Note above, we can compute $\mathrm{Ext}^i_P(\mathcal{F}, \omega_P)$ as the cohomology of the complex $\mathrm{Hom}_X(\mathcal{F}, \mathcal{J}^\bullet)$, or of $\mathrm{Hom}_X(\mathcal{F}, \mathcal{J}_2^\bullet)$. For $i = r$,

$$\ker\Big(\mathrm{Hom}_X(\mathcal{F}, \mathcal{J}_2^r) \to \mathrm{Hom}_X(\mathcal{F}, \mathcal{J}_2^{r+1})\Big) \cong \mathrm{Hom}_X(\mathcal{F}, \ker(\mathcal{J}_2^r \to \mathcal{J}_2^{r+1})) \cong \mathrm{Hom}_X(\mathcal{F}, \omega_X^\circ).$$

□

Proposition 13. [H, III.7.5] *Let X be projective over a field $\mathbf{k}$. Then X has a dualizing sheaf.*

Proof. Embed it in $P = \mathbb{P}^N_{\mathbf{k}}$ with codimension r, and let $\omega_X^\circ = \mathcal{E}xt^r_P(\mathcal{O}_X, \omega_P)$. We just proved that for any $\mathcal{O}_X$-module $\mathcal{F}$,

$$\mathrm{Hom}_X(\mathcal{F}, \omega_X^\circ) \cong \mathrm{Ext}^r_P(\mathcal{F}, \omega_P).$$

If $\mathcal{F}$ is coherent, duality for P says

$$\mathrm{Ext}^r_P(\mathcal{F}, \omega_P) \cong H^{N-r}(P, \mathcal{F})^* \cong H^{\dim X}(X, \mathcal{F})^*.$$

Hence $\mathrm{Hom}_X(\mathcal{F}, \omega_X^\circ) \cong H^{\dim X}(X, \mathcal{F})^*$. If we plug in $\mathcal{F} = \omega_X^\circ$), the identity goes to some element of $H^{\dim X}(\omega_X^\circ, \mathcal{F})^*$, which we take for our trace morphism t. □

Lemma 12. *Let $y \in X \subseteq P$ be a regular point of both X and P, defined over an algebraically closed field $\mathbf{k}$. Then there is a projective resolution of y's local ring in X as a quotient of y's local ring in P whose length = the codimension of X in P.*

Proof. Cohen's structure theorem, which we won't prove, says that local ring at a smooth point is a power series ring in $\mathbf{k}$. □

Theorem 15. [H, III.7.6] *Let* X *be dimension* n, *and projective over* $\mathbf{k}$ *algebraically closed. Let* ω_X° *be a dualizing sheaf, and* $\mathcal{O}(1)$ *very ample.*

(a) *For all coherent* $\mathcal{F}$, *there are functorial* maps
$$\theta^i : \mathrm{Ext}^i(\mathcal{F}, \omega_X^\circ) \to H^{n-i}(X, \mathcal{F})^*,$$
such that θ^0 *is the one defined using* t.

(b) *if* X *is regular, then those maps are isomorphisms.*

(The converse of (b) is not true – that only implies that X is "Cohen-Macaulay".)

Proof. (a) Let $\mathcal{F}$ be a quotient of $\mathcal{E} := \oplus_{j=1}^N \mathcal{O}(-q << 0)$. Then
$$\begin{aligned} \mathrm{Ext}^i(\mathcal{E}, \omega_X^\circ) &\cong \oplus_{j=1}^N \mathrm{Ext}^i(\mathcal{O}(-q), \omega_X^\circ) \\ &\cong \oplus_{j=1}^N \mathrm{Ext}^i(\mathcal{O}, \omega_X^\circ(q)) \\ &\cong \oplus_{j=1}^N H^i(\omega_X^\circ(q)) \\ &= 0 \qquad \text{for } i > 0 \text{ and } q >> 0. \end{aligned}$$
Thus the LHS is a coeffaceable contravariant δ-functor, so universal, so it maps to the RHS since that's a contravariant δ-functor with a given isomorphism in degree 0.

(b) To get these to be isomorphisms, we need the $H^{n-i}(X, \bullet)^*$ to give a universal δ-functor too, and it's enough for it to be coeffaceable. If we can show that $H^{n-i}(X, \mathcal{E})^* = 0$ for $i > 0$, or equivalently $H^{n-i}(X, \mathcal{O}(-q))^* = 0$, we're done.

Let $P = \mathbb{P}_{\mathbf{k}}^N$ be the projective space X is embedded in. Then using duality for P,
$$\begin{aligned} H^{n-i}(X, \mathcal{O}(-q)) &\cong H^{n-i}(P, \mathcal{O}_X(-q)) \\ &\cong \mathrm{Ext}_P^{N-(n-i)}(\mathcal{O}_X(-q), \omega_P) \\ &\cong \mathrm{Ext}_P^{N-(n-i)}(\mathcal{O}_X, \omega_P(q)) \\ &\cong \Gamma(P; \mathcal{E}xt_P^{N-(n-i)}(\mathcal{O}_X, \omega_P(q))) \end{aligned}$$
We will actually show $\mathcal{E}xt_P^{N-(n-i)}(\mathcal{O}_X, \bullet) = 0$ for $i > 0$. Look at the stalk at $y \in X$:
$$\mathcal{E}xt_P^{N-(n-i)}(\mathcal{O}_X, \mathcal{G})|_y \cong \mathrm{Ext}_{\mathcal{O}_{y \in P}}^{N-(n-i)}(\mathcal{O}_{y \in X}, \mathcal{G}_y)$$
We can compute this RHS using a projective resolution of the local ring $\mathcal{O}_{y \in X}$ as a quotient of the local ring $\mathcal{O}_{y \in P}$. Because y is a regular point of X, and $\mathbf{k}$ is algebraically closed, there is a projective resolution of length = the codimension $N - n$ (lemma 12). □

Corollary 6. *Let* X *be smooth and projective over* $\mathbf{k}$, *algebraically closed. Then for* $\mathcal{F}$ *locally free,*
$$H^i(X, \mathcal{F}) \cong H^{n-i}(X, \mathcal{F}^* \otimes \omega_X^\circ)^*.$$

Proof.
$$H^i(X, \mathcal{F}) \cong \mathrm{Ext}^i(\mathcal{O}_X, \mathcal{F}) \cong \mathrm{Ext}^i(\mathcal{F}^* \otimes \omega_X^\circ, \omega_X^\circ) \cong H^{n-i}(X, \mathcal{F}^* \otimes \omega_X^\circ)^*.$$
□

Theorem 16 (III.7.11)**.** *Let* $X \subseteq \mathbb{P}_{\mathbf{k}}^N$ *be a smooth projective variety of codimension* r, *with ideal sheaf* $\mathcal{I}$. *Then* $\omega_X^\circ \cong \omega_P \otimes (\mathcal{I}/\mathcal{I}^2) \cong \omega_X$.

Proof. This proof is a nasty local calculation, using a sequence locally defining X inside $\mathbb{P}^N$ to give a projective resolution of $\mathcal{O}_X$. □

14. Higher Direct Images of Sheaves

Given a map $f : X \to Y$, we have the functors $f_* : \mathfrak{Ab}(X) \to \mathfrak{Ab}(Y)$ and $f_* : \mathbf{Mod}(X) \to \mathbf{Mod}(Y)$. If Y is a point, $f_*(\bullet) = \Gamma(X, \bullet)$, which we've spent lots of time looking at the right derived functors of. So let's do the same for any f, defining $R^i f_* : \mathfrak{Ab}(X) \to \mathfrak{Ab}(Y)$. Most of the results about it should look very familiar, which I suppose could be viewed as boring; I see it as comforting.

Here is an alternate way to motivate the naturality of these sheaves:

Proposition 14. [H, III.8.1] *The sheaf* $R^i f_*$ *is associated to the presheaf*

$$(V \subseteq Y) \mapsto H^i(f^{-1}(V), \mathcal{F}|_{f^{-1}(V)}).$$

Proof. To show that some list of functors forms the higher derived functors of a given one, use the hammer: show it's a universal δ-functor, by showing it's effaceable.

How is the RHS a δ-functor at all? Since the H^i form long exact sequences, the presheaves fit into long exact sequences, and the functor presheaf $\mapsto$ associated sheaf is exact, so the sheaves fit into long exact sequences too.

For $i = 0$, this is just $f_*\mathcal{F}$, by definition. If $\mathcal{I} \in \mathfrak{Ab}(X)$ is an injective sheaf, then it remains injective when restricted to open sets (a nontrivial fact which we proved, [H, III.6.1]), so the RHS vanishes for such $\mathcal{I}$. □

Corollary 7. [H, III.8.2-3]

1. $R^i f_*$ *can be computed locally, i.e.* $R^i f_*(\mathcal{F})|_V = R^i f'_*(\mathcal{F}|_{f^{-1}(V)})$ *where* $f' : f^{-1}(V) \to V$ *is the restriction.*
2. *If* $\mathcal{F}$ *is flasque, then* $R^i f_*(\mathcal{F}) = 0$ *for* $i > 0$.

Proof.
(1) The RHS of the proposition above is obviously local.
(2) The restriction of flasque to open sets is flasque, and flasque sheaves have no higher H^i, so the RHS is zero.

□

The $R^i f_*$ were defined as right derived functors of the functor between categories of sheaves, not $\mathcal{O}$-modules, so one might worry that that would require separate notation, but fear not:

Proposition 15. [H, III.8.4] *Let* $f : X \to Y$ *be a morphism of ringed spaces. Then the right derived functors of* $f_* : \mathbf{Mod}(X) \to \mathbf{Mod}(Y)$, *followed by* $\mathbf{Mod}(Y) \to \mathfrak{Ab}(Y)$, *are the* $R^i f_*$.

Proof. We can use the same resolution to compute the derived functors of the two functors: take a resolution by injective $\mathcal{O}_X$-modules, which are flasque, hence acyclic for the $\mathfrak{Ab}()$ functor, so can be used to compute those derived functors too. □

One can nicely mix "sheaf cohomology is about pushing to a point" with "higher sheaf cohomology of quasicoherents on affine varieties vanishes":

Proposition 16. [H, III.8.5] *Let X be Noetherian, $f : X \to Y = \operatorname{Spec} A$ a morphism, $\mathcal{F}$ quasicoherent on X. Then $R^i f_*(\mathcal{F})$ is the sheaf associated to the A-module $H^i(X;\mathcal{F})$.*

Proof. Usual technique: show the second list forms a universal δ-functor by showing it's effaceable, and that it's right for $i = 0$.

For $i = 0$, $H^i(X;\mathcal{F}) = \Gamma(\mathcal{F}) = f_*(\mathcal{F})$. So that's clear.

The functor $\tilde{}$ taking an A-module to its sheaf on Y is exact, so the second list is a δ-functor. We can embed $\mathcal{F}$ into a flasque quasicoherent sheaf, with which to efface said δ-functor. □

Corollary 8. [H, III.8.6] *Let $f : X \to Y$, X Noetherian, $\mathcal{F}$ quasicoherent on X. Then $R^i f_*(\mathcal{F})$ is quasicoherent on Y.*

Proof. Use the locality of $R^i f_*$ to check it on affine opens in Y. □

Theorem 17. [H, III.8.8] *Let $f : X \to Y$ be a projective morphism of Noetherian schemes, let $\mathcal{O}(1)$ be a very ample invertible sheaf on X over Y, and let $\mathcal{F}$ be coherent on X.*

(1) *For $i \geq 0$, $R^i f_*(\mathcal{F})$ is coherent on Y.*
(2) *For $i > 0$, $n >> 0$, $R^i f_*(\mathcal{F}(n)) = 0$.*

Even if f isn't projective, $R^i f_*(\mathcal{F}) = 0$ for $i > \dim X$.

Proof. Since the questions are local on Y, we can assume it's affine. Then $R^i f_*(\mathcal{F}) \cong H^i(X;\mathcal{F})$, and we proved these Serre and Grothendieck theorems already. □

Recall the definition of K-homology of a scheme, as made from (isomorphism classes of) coherent sheaves modulo (short) exact sequences.

Theorem 18. *Let $f : X \to Y$ be a projective morphism of Noetherian schemes, and define $f_* : K_0(X) \to K_0(Y)$ by*

$$f_*([\mathcal{F}]) := \sum_{i=0}^{\dim X} (-1)^i [R^i f_*(\mathcal{F})].$$

Then f_* is well-defined.

In particular, if Y is a point, then $\sum_{i=0}^{\dim X} (-1)^i [H^i(X;\mathcal{F})]$ is a well-defined integer, sometimes called the "holomorphic Euler characteristic".

Proof. If $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ is a short exact sequence of coherent sheaves, we get a long exact sequence

$$0 \to f_*(\mathcal{A}) \to f_*(\mathcal{B}) \to f_*(\mathcal{C}) \to R^1 f_*(\mathcal{A}) \to \ldots \to R^{\dim X} f_*(\mathcal{C}) \to 0.$$

This long exact sequence of coherent sheaves gives an equation in $K_0(Y)$:

$$f_*([\mathcal{A}]) - f_*([\mathcal{A}]) + f_*([\mathcal{C}]) = 0$$

which is exactly what we wanted to show. □

How to compute it?

Theorem 19. *Let $f : X \to Y$ be a projective morphism of Noetherian schemes, let $\mathcal{O}(1)$ be a very ample invertible sheaf on X over Y, and let $\mathcal{F}$ be coherent on X.*

Then $f_*([\mathcal{F}])$ can be computed as follows: $f_*([\mathcal{F}(n)])$ is a polynomial in n (with coefficients in $K_0(Y)$), which for large n is $[f_*(\mathcal{F}(n))]$, and for $n = 0$ is $f_*([\mathcal{F}])$.

Moreover, the degree of this polynomial is $\dim \mathrm{supp}(\mathcal{F})$.

Proof. Once we prove it's a polynomial, the Serre vanishing above proves the second statement, and the third is true by definition.

Let $g(n) = f_*([\mathcal{O}(n)])$, and consider $\Delta g(n) = g(n+1) - g(n)$, which is a polynomial iff g is.

Let $b \in \Gamma(\mathcal{O}(1))$ be a function not vanishing on any component of $\mathrm{supp}(\mathcal{F})$. Then $\cdot b : \mathcal{F} \to \mathcal{F}(1)$ is injective, and defines a short exact sequence $0 \to \mathcal{F} \to \mathcal{F}(1) \to \mathcal{E} \to 0$. Twisting that by $\mathcal{O}(n)$, we get $\Delta g(n) = f_*([\mathcal{E}(n)])$. Since $\dim \mathrm{supp}(\mathcal{E}) = \dim \mathrm{supp}(\mathcal{F}) - 1$, we can use induction to assert that Δg is a polynomial of degree $\dim \mathrm{supp}(\mathcal{E}) = \dim \mathrm{supp}(\mathcal{F}) - 1$.

For the base case, $\mathcal{F}$ is supported on a bunch of points, and $\mathcal{E}$ is empty. Hence the short exact sequence says that $\mathcal{F}(n) \cong \mathcal{F}(n+1)$, so g is constant. Then we need to be sure that that constant is not 0. Since $\mathrm{supp}\mathcal{F}$ is finite hence affine, $\mathcal{F} \cong \Gamma(\mathrm{supp}\mathcal{F}, \mathcal{F})\tilde{}$, and $\Gamma(\mathrm{supp}\mathcal{F}, \mathcal{F})$ being 0 would imply $\mathrm{supp}\mathcal{F} = 0$, contradiction. □

15. Higher direct images: an example

Let X be the blowup of $\mathbb{A}^3_{\mathbf{k}}$ at the origin, and $Y \subseteq X$ the proper transform of the coordinate planes. In coordinates,

$$X = \mathrm{Proj}\, \mathbf{k}[x^{(0)}, y^{(0)}, z^{(0)}, a^{(1)}, b^{(1)}, c^{(1)}] / \langle \text{"}a/b = x/y, a/c = x/z, b/c = y/z\text{"} \rangle$$

$$I_Y = \langle abc, xyz \rangle$$

This X has an affine open cover $U_a \cup U_b \cup U_c$, where $U_i = \{i \neq 0\}$. Let $V_i = U_i \cap Y$, so e.g.

$$V_c = \mathrm{Spec}\, \mathbf{k}[x, y, z, a', b'] / \langle a'y = b'x, a'z = x, b'z = y, a'b', xyz, a'b' \rangle \cong \mathrm{Spec}\, \mathbf{k}[z, a', b'] / \langle a'b' \rangle$$

where $a' = a/c, b' = b/c$. Also,

$$V_c \cap V_b \cong \mathrm{Spec}\, \mathbf{k}[z, a', b'^{\pm}] / \langle a'b' \rangle \cong \mathrm{Spec}\, \mathbf{k}[z, b'^{\pm}]$$

and $V_a \cap V_b \cap V_c$ is empty, since $abc = 0$.

Let $\pi : X \to \mathbb{A}^3_{\mathbf{k}}$ be the blowdown map. We compute $R^i\pi_*\mathcal{O}_Y$, $i \geq 0$.

- Since the target is affine and $\mathcal{O}_Y$ is a quasicoherent sheaf on X, we just need to compute $H^i(X, \mathcal{O}_Y)$.
- That's isomorphic to $H^i(Y, \mathcal{O}_Y)$.
- When we compute this with Čech cohomology, the groups are just rings of functions, since the sheaf is the structure sheaf.

$$0 \to \mathbf{k}[V_a] \oplus \mathbf{k}[V_b] \oplus \mathbf{k}[V_c] \to \mathbf{k}[V_a \cap V_b] \oplus \mathbf{k}[V_a \cap V_c] \oplus \mathbf{k}[V_b \cap V_c] \to 0$$

If we make these $\mathbb{Z}^2$-graded modules, by giving x, y, z degrees $(1,0), (0,1), (-1,-1)$ and a, b, c the same, then these modules are supported in ...

16. Flat morphisms

Let A be a commutative ring, and M an A-module, so $\tilde{M}$ is a sheaf on $\operatorname{Spec} A$. Then $M \otimes_A \bullet$ is a covariant endofunctor of $\operatorname{Mod}(A)$; call M **flat** if $M \otimes_A \bullet$ is an exact functor. This turns out to be a handy notion to say that $\tilde{M}$ doesn't jump around so much from point to point in $\operatorname{Spec} A$.

Our basic non-example: $A_{non} = \mathbf{k}[x]$, $M_{non} = A_{non}/\langle x \rangle$, in which the 0 fiber is bigger than other fibers.

Proposition 17. [H, III.9.1]

(1) *M is flat iff for all f.g. ideals I, $I \otimes_A M \to M$ is injective. (Non-example: $I = \langle x \rangle$.)*
(2) *If M flat over A, and $A \to B$, then $M \otimes_A B$ is flat over B. ("Pullback of a flat family is flat.")*
(3) *Transitivity: if B is a flat A-algebra, and N flat over B, then N is flat over A.*
(4) *M is flat over A iff M_p is flat over A_p for all $p \in \operatorname{Spec} A$.*
(5) *Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a SES. Then M_1, M_3 flat implies M_2 flat, and M_2 and M_3 flat implies M_1 flat.*
(6) *If A is local Noetherian, and M f.g., then M flat iff M free.*
(7) *Projective modules (or more generally direct summands of flat modules) are flat.*

Proof. Matsumura [2,Ch.2,§3], except the last one. For that, write $F = M \oplus M'$, where F is free hence flat. Then when we tensor a SES with F, we get a direct sum of two complexes, whose total cohomology is zero, hence each one has zero cohomology, so is again exact. $\square$

Proposition 18. (1) *Flat implies torsion-free. If A is a PID, then they are equivalent. If in addition M is f.g., then they imply free.*
(2) *The restriction to an open set $\operatorname{Spec} S^{-1}A$ of a flat module is again flat. Hence it's clear how to extend the notion of flatness to sheaves over general scemes.*

Proof. (1) If M is flat, then $\langle t \rangle \otimes_A M \to M$ is injective, hence M is torsion-free. In a PID, every ideal looks like this. If M is a f.g. module over a PID, torsion-free implies free.
(2) Since we can check flatness at local rings, this is clear.

$\square$

At this point one should rephrase [H, III.9.1] in sheaf language, e.g. "free" gets replaced by "locally free", and obtain [H, III.9.2].

Non-example: let $Y = \operatorname{Spec} \mathbf{k}[x,y]/\langle xy \rangle$, and X its normalization $\operatorname{Spec}(\mathbf{k}[x] \oplus \mathbf{k}[y])$. Then X is not flat over Y; since it is coherent and rank 1, if locally free it should be invertible, but it needs two generators over the origin.

My favorite flatness situation: $Y = \operatorname{Spec} \mathbf{k}[t]$, and X is projective over Y, say $X \subseteq \operatorname{Proj} \mathbf{k}[t, x_0^{(1)}, \ldots, x_N^{(1)}]$. Then $\pi_*(\mathcal{O}_X)$ is a graded module over Y, so torsion-free iff each graded component is torsion-free. The graded components are finitely generated modules over the PID $\mathbf{k}[t]$, so torsion-free iff they are actually free! If we look at the $t = k$ fiber (by modding out by $\langle t - k \rangle$), we get a homogeneous ideal in $\mathbf{k}[x_0^{(1)}, \ldots, x_N^{(1)}]$ whose graded codimension is the **Hilbert function**. Then flatness says that every fiber has the same Hilbert function.

Say $\mathcal{F}$ is a sheaf on X, that we want to push down along $f : X \to Y$. Say we extend the base with $u_Y : Y' \to Y$, and get a new family $f' : X' \to Y'$, where X' is the pullback and $u_X : X' \to X$. Now let's try to push the sheaf $\mathcal{F}'$ down along $X' \to Y'$.

Proposition 19. [H, III.9.3] *Say $\mathcal{F}$ is quasicoherent, and $f : X \to Y$ is separated of finite type. If u_Y is flat, then we can compute higher direct images before and after, i.e.* $u^* R^i f_*(\mathcal{F}) \cong R^i f'_*(u_X^* \mathcal{F})$.

Proof. The question is local on Y,Y′, so assume they're both affine, $Y = \operatorname{Spec} A$, $Y' = \operatorname{Spec} A'$. Then the higher direct images are just cohomology groups (there's quasicoherence used), and the goal is to prove

$$H^i(X, \mathcal{F}) \otimes_A A' \cong H^i(X', \mathcal{F}').$$

We can compute the $H^i(X, \mathcal{F})$ by Čech cohomology, and get a complex of A-modules. If we tensor with A', we get the corresponding Čech complex to compute $H^i(X', \mathcal{F}')$ (this uses separatedness to infer the open affine cover). But we want to take cohomology first, *then* tensor with A'; what says that these commute is that Y' be flat over Y. □

It seems needlessly general, but one can define $\mathcal{F}$ on X to be **flat over** Y **at** $x \in X$, via $f : X \to Y$, if $\mathcal{F}_x$ is a flat $\mathcal{O}_{f(x),Y}$-module. Then say the morphism $f : X \to Y$ is **flat at** $y \in Y$ if $\mathcal{O}_X$ is flat at every $x \in f^{-1}(y)$.

We will very often be interested in how these fibers $f^{-1}(y)$ vary, and when we do so we'll call f a **flat family**, which means nothing more than "flat morphism". If X is a closed subscheme of $P \times Y$, with f the restriction to X of the projection to the second factor, then each $f^{-1}(y)$ is naturally a subscheme of P. In this case we will call f a **flat family of subschemes of** P, or if P is projective space, a **flat family of projective schemes**. Notice that the condition X *closed* in $P \times Y$ means slightly more than f being flat and its fibers being subschemes of P.

Non-example: let $Y = \operatorname{Spec} \mathbf{k}[t]$, $P = \operatorname{Spec} \mathbf{k}[x, y, z]$, and X the subfamily define by $I = \langle x - ty, x - tz \rangle$. The fiber over $t \neq 0$ is a line, and over $t = 0$ is a plane, which seems bad. And indeed, $y - z$ is a torsion element of $\mathbf{k}[t, x, y, z]/I$, killed by $\langle t \rangle$, so this family is not flat. We can fix it by excising the irreducible component of X lying over $t = 0$, replacing I by $I : \langle t \rangle$. One bit of language used in this context is "$x - ty, x - tz$ is not a Gröbner basis".

16.1. **Rees families.** Here is a basic example of flat families. Let A be a commutative $\mathbf{k}$-algbebra with a decreasing filtration $A = A_0 \supset A_1 \supset A_2 \supset \ldots$, meaning that $A_i A_j \subseteq A_{i+j}$. So each A_k is an ideal, containing A_1^k, and a particularly special case is $A_k = A_1^k \forall k$, the A_1**-adic filtration**. Let $A_n := A$ for $n \leq 0$, and define the $\mathbf{k}[t]$-subalgebra

$$Rees(A_\bullet) := \oplus_{n \in \mathbb{Z}} A_n t^{-n} \subseteq A[t, t^{-1}].$$

Since $A[t, t^{-1}]$ is a torsion-free $\mathbf{k}[t]$-module, so is $Rees(A_\bullet)$, hence it is flat. The fibers of $\operatorname{Spec} Ress(A_\bullet)$ to the line $\operatorname{Spec} \mathbf{k}[t]$ over elements of $\mathbf{k}^\times$ are isomorphic to $\operatorname{Spec} A$. But the fiber over $t = 0$ is $\operatorname{Spec} \operatorname{gr} A$, where $\operatorname{gr} A := \oplus_{i \in \mathbb{N}} R_i / R_{i+1}$.

For example, let $A = \mathbf{k}[x, y, z]/\langle xz - y^2 \rangle$, which embeds in $\mathbf{k}[a, b]$ as the even-degrees part, $(x, y, z) \mapsto (a^2, ab, b^2)$. So $\operatorname{Proj} A \cong \mathbb{P}^1_{\mathbf{k}}$. If we $\langle y \rangle$-adically filter, then $\operatorname{gr}_y A \cong \mathbf{k}[x, y, z]/\langle xz \rangle$, whose Proj is $\mathbb{P}^1 \cup_{\mathbb{P}^0} \mathbb{P}^1$, so flat families can contain irreducible and reducible fibers. If we $\langle x \rangle$-adically filter, then $\operatorname{gr}_y A \cong \mathbf{k}[x, y, z]/\langle y^2 \rangle$, whose Proj is a double line, so flat families can contain reduced and nonreduced fibers.

Here is the most basic property of flatness:

Proposition 20. [H, III.9.5] *Let $f : X \to Y$ be a flat morphism of schemes of finite type over a field $\mathbf{k}$. Then every fiber has the expected dimension.*

Proof. What dimension should we expect? X and Y have not been assumed equidimensional. Really, we should look nearby a point $x \in X$, mapping to $y \in Y$, at the neighborhood of x in $f^{-1}(y)$. Then the expectation is

$$\dim_x(f^{-1}(y)) = \dim_x X - \dim_y Y$$

where those are the dimensions of the local rings.

Okay, first replace Y by the formal neighborhood $Y' = \operatorname{Spec} \mathcal{O}_{y,Y}$ of y, and X by $X' = X \times_Y Y'$. This is again flat by Base Change. So we can assume y is a closed point, $\dim Y = \dim_y Y$, Y affine... We can also replace Y by its reduction without changing anything.

If $\dim Y = 0$, then everything is trivial.

If $\dim Y > 0$, pick a nonzero divisor t in the maximal ideal, and consider the base change $Y' := \{t = 0\} \hookrightarrow Y$. This is codim 1 by the assumption $t \not| 0$. We finally use f flat to infer that f^*t is also not a zero divisor. Hence $f^{-1}(Y')$ is also codim 1 in X. The LHS doesn't change, so

$$\dim_x(f^{-1}(y)) = \dim_x X' - \dim_y Y' = (\dim_x X - 1) - (\dim_y Y - 1) = \dim_x X - \dim_y Y$$

by induction. ☐

Flatness is easy to check over normal (so, regular) 1-d bases; loosely speaking, every component should spread out over the base (and this is exactly correct if X is reduced). If X is not reduced, we need to generalize our notion of "component" a bit. Say that $x \in \operatorname{Spec} X$ is an **associated point** (probably not a closed point) if the maximal ideal $\mathfrak{m}_x \mathcal{O}_{x,X}$ consists of zero divisors. For example, if $X = \operatorname{Spec} \mathbf{k}[a, b]/\langle ab, b^2 \rangle$ so $\mathcal{O}_{x,X} = \mathbf{k}[[a, b]]/\langle ab, b^2 \rangle$ and $\mathfrak{m}_x = \langle a, b \rangle$, then $bi = 0$ for all $i \in \mathfrak{m}_x$.

Stupider case: if x is the generic point of a geometric component of X, then $\mathcal{O}_{x,X}$ is dimension 0, hence Artinian, so $\mathfrak{m}_x$ consists of nilpotents. This is the only possibility, if X is reduced. Other associated points are called **embedded components**.

Proposition 21. [H, III.9.7] *Let $f : X \to Y$, where Y is regular of dim 1 and X is reduced. Then f is flat iff every associated point of X dominates Y, rather than lying over some closed point.*

Proof. Say f is flat, $y = f(x)$ is a closed point, and t is a uniformizing parameter of the DVR $\mathcal{O}_{y,Y}$. So it's not a zero divisor, and $\mathcal{O}_{x,X}$ is flat hence torsion-free over this PID, so $f^*t \in \mathfrak{m}_x$ is not a zero divisor, hence x is not an associated point.

Conversely, say every associated point maps to the generic point of Y. We want to show f is flat over every local ring $\mathcal{O}_{y,Y}$ which is trivial if y is the generic point (since $\mathcal{O}_{y,Y}$ is a field), so assume y is a closed point and base change to $\mathcal{O}_{y,Y}$.

Let $f(x) = y$. We want to show $\mathcal{O}_{x,X}$ is a flat module over the DVR $\mathcal{O}_{y,Y}$, or equivalently that it's torsion-free. Otherwise f^*t is a zero divisor, whose annihilator is some ideal in $\mathcal{O}_x$ each of whose components gives an associated point mapping to y. ☐

If X is Noetherian, it has finitely many associated points, hence finitely many $y \in Y$ over which f is not flat. If we excise those fibers (by taking the closure of the complement), we get a flat family.

Let Y be regular of dimension 1, $Y^\times := Y \setminus \{y\}$, and $F^\times \subseteq \mathbb{P}^n_{Y^\times}$ be a family over $Y^\times$. (Maybe flat, by excising bad fibers as above.) Then we can define a family $F \subseteq \mathbb{P}^n_Y$ by taking the closure of $F^\times$ inside $\mathbb{P}^n_Y$, and it will be automatically flat over $y \in Y$ by the above theorem. The fiber F_y is called the **limit subscheme** of the family $F^\times$.

Special example: let $F_1 \subseteq \mathbb{P}^n_{\mathbf{k}}$, and let $\mathbb{G}_m$ act on $\mathbb{P}^n_{\mathbf{k}}$ by $z \cdot (c_0, \ldots, c_n) := (z^{e_0} c_1, \ldots, z^{e_n} c_n)$ for some $\vec{e} \in \mathbb{Z}^{n+1}$. Define $F^\times \subseteq \mathbb{P}^n_{Y^\times}$, where $Y = \operatorname{Spec} \mathbf{k}[t]$ and $y = \{0\}$, by

$$F^\times = \{(p, t) : t^{-1} \cdot p \in F_1\}.$$

Then $F^\times$ is automatically flat, and F_0 is called the **initial scheme** of F_1 with respect to $\vec{e}$. (This $\vec{e}$ defines a partial order on monomials, and F_0 is defined by the ideal spanned by the initial terms of F_1's ideal. Exercise: connect this to standard descriptions of Gröbner basis theory.)

Note that if $F^\times$ is just a projective-able, not actually projectived, family over Y, then there is no uniquely defined limit subscheme. Let $\mathbb{P}^1_Y := \operatorname{Proj} \mathbf{k}[t^{(0)}, x^{(1)}, y^{(1)}]$, and consider the two families defined by $x^2 = ty^2$ and $y(y - x) = 0$, which are isomorphic away from $t = 0$ but have different fibers over $t = 0$.

One way of saying this is that the functor

$Y \mapsto$ isomorphism classes of flat families with base Y, whose fibers are projective schemes

is not "separated". To make sense of this, one transfers the definition for schemes over to the representable functors they represent, and then notices that one can phrase it as a definition about functors.

Theorem 20. *Fix* N.

(1) *The functor*

$$Y \mapsto \textit{closed subschemes of } \mathbb{P}^d_Y\textit{, flat over } Y$$

is complete and separated. If representable by a scheme (necessarily complete and separated), the scheme is only of finite type if one fixes the Hilbert polynomial, which we do hereafter.

(2) *(Grothendieck) It is indeed representable, by what we call the* **Hilbert scheme** *of that Hilbert polynomial.*

(3) *(Mumford) Hilbert schemes are projective.*

(4) *(Hartshorne) Hilbert schemes are connected.*

(5) *(Fogarty) The Hilbert scheme of "n points in $\mathbb{P}^2$" ($d = 2$, $h \equiv n$) is smooth and irreducible.*

(6) *(Vakil) Hilbert schemes of curves in $\mathbb{P}^3$ already contain every local singularity over $\mathbb{Z}$.*

Proof. (1) For "complete", we need to construct limit subschemes, which we can do by taking closures. For "separated", we need to use our characterization of flatness to say that this is the only possible construction.

(2)

(3) Mumford shows that if one only considers saturated homogeneous ideals (defining these projective schemes), there is a large N at which the Hilbert function necessarily matches the Hilbert polynomial. Hence one obtains an embedding of the Hilbert scheme into a certain finite-dimensional Grassmannian.

(4) Using Gröbner bases, one can connect any subscheme with a monomial subscheme. Then one has to "distract" these subschemes to kick them all downhill into a fixed particular one.

(5) The tangent space at $I \leq R$ is $\mathrm{Hom}_R(I, R/I)$, much like the tangent space to the Grassmannian. The dimension of tangent space is a semicontinuous function, i.e. to show smoothness you only need to check at the worst possible points. This is not hard to do in this case.

(6)

□

The fact that the Hilbert scheme of n points in the plane is so much better behaved than most Hilbert schemes suggests, perhaps, that its appearance here is sort of accidental, and it is better to see it as a member of an entirely different family. That family is the family of Grojnowski-Nakajima quiver varieties, about which we'll say much more in the fall.

Perhaps it seems sad that a family of reduced subschemes might have a limit that is nonreduced. Valery Alexeev and I showed that if one is willing to replace subschemes $F_0 \subseteq \mathbb{P}^n$ by finite morphisms $F_0 \to \mathbb{P}^n$ of reduced subschemes, which we called **branchvarieties of** $\mathbb{P}^n$, then limit branchvarieties are unique. One doesn't quite have existence; it may require a finite base change, which is pretty much good enough.

Here's an example of the difference between subschemes and branchvarieties, in $\mathbb{P}^3 = \mathrm{Proj}\, \mathbf{k}[x, y, z, w]$. Consider the family defined by $\langle z, w\rangle \cap \langle x, ty - w\rangle$, for $t \neq 0$. To compute the limit scheme as these two skew lines collide, first compute the intersection

$$\langle xz, wx, (ty - w)z, (ty - w)w\rangle$$

then add $\langle t\rangle$, to get $\langle t, xz, wz, wx, w^2\rangle$ which is one dimension larger than its radical, $\langle t, xz, w\rangle = \langle t\rangle + (\langle z, w\rangle \cap \langle x, w\rangle)$. So the limit scheme is not the union of the two limit lines, but has one embedded point, at the crossing.

That nonreduced scheme can also be obtained as a limit of a point falling into the crossing. Put backwards, the nonreduced scheme can be deformed to a reduced scheme by letting the embedded point evaporate off. In this way, if one wants to study equidimensional schemes – e.g. irreducible varieties – the Hilbert scheme forces one to buy non-equidimensional varieties as part of the package deal. (One can't blame the reducibility of the two lines, either – exactly the same happens if we squash a space curve into the plane and make it self-intersect, as in [H, p260].)

This unpleasant behavior, at least, does not occur for branchvarieties; we prove that in a flat family of branchvarieties over a connected base, if one fiber is equidimensional then all are.

17. Smooth morphisms

With two classes left, I'm just looking to point out the highlights of the remainder of [H].

A "smooth morphism" is what you would come up with if you wanted to generalize the notion of "X is smooth" to "X is smooth over a base", something more like "every fiber is smooth". In particular, most algebraic maps $X \to Y$ between smooth varieties *are not smooth morphisms!!!*

Note that even the usual notion, in terms of dimensions of tangent spaces, has a weirdness: is $\mathbb{P}^1 \coprod \mathbb{P}^2$ smooth? To sidestep this we define $f : X \to Y$ **smooth of relative dimension** n if

(1) it's flat,
(2) if a component X' of X hits a component Y' of Y, $\dim X' = \dim Y' + n$, and
(3) over any point x, $\dim_{k(x)}(\Omega_{X/Y} \otimes k(x)) = n$.

This last should be something like the "cotangent spaces to the fiber".

It's entirely possible that f is smooth without X or Y being smooth. Dumb case: $X = Y \times Z$ and f is a projection. Superdumb case: $X = Y$ and f is the identity.

17.1. **Characteristic p weirdness.**

Trickier example: suppose that $\mathbf{k}$ is a field of characteristic $p > 2$ and that a is an element of $\mathbf{k}$ which does not have a pth root in $\mathbf{k}$. Then

$$\operatorname{Spec} \mathbf{k}[x, y]/\langle x^2 - y^p + a \rangle$$

is a regular scheme, but not smooth over $\operatorname{Spec} \mathbf{k}$.

The point is that "smooth" is set up to behave well under base change, whereas regular evidently isn't. If we let b be a pth root of a in $\bar{\mathbf{k}}$, then when we extend to $\bar{\mathbf{k}}[x, y]/\langle x^2 - y^p + a \rangle$, it is no longer regular.

(How did this happen? In either case the nonregular locus is given by $(2x, 0) = (0, 0)$, but that line misses the curve if $y^p = a$ has no solution.)

17.2. **Characteristic 0 goodness.**

The principal surprising result about smoothness is that if Y is defined over a field of characteristic 0, then it contains an open set over which f is smooth, a sort of algebraic Sard's theorem. (If f is the Frobenius endomorphism of $\mathbb{P}^1_{\mathbb{F}_p}$, there is no such open set.)

Kleiman used this in 1973 to give a proof of Bertini's theorem [H, III.10.9]: if X is a nonsingular projective variety over an algebraically closed field of characteristic 0, then "almost every" hyperplane section of X is also nonsingular.

There are whole books of "Bertini theorems", many of which hold in characteristic p too, taking good properties of X to good properties of generic hyperplane sections.

18. The Theorem on Formal Functions

The main theorem here is technical to state and more technical to prove. If $\mathcal{F}$ is a sheaf on X, $f : X \to Y$ is a projective morphism, and $y \in Y$ a point, then we can compute three things about $R^i f_*(\mathcal{F})$ near y:

(1) The inverse limit, of restricting $R_i f_*(\mathcal{F})$ to infinitesimal neighborhoods of $y \in Y$;
(2) The inverse limit of $H^i(X_n, \mathcal{F}_n)$, where $X_n = f^{-1}$ of the nth infinitesimal neighborhood;
(3) $H^i(\hat{X}, \hat{\mathcal{F}})$ where $\hat{X}$ is the formal neighborhood of X_y.

Hartshorne proves that the first two are isomorphic.

A simple consequence: if the fibers have dimension $\leq r$, then $R^i f_*(\mathcal{F}) = 0$ for $i > r$ and $\mathcal{F}$ coherent. (Proof: use the second formula to give the first.)

Theorem 21. [H, III.11.3-5]

(1) *Let $f : X \to Y$ projective take $f_*\mathcal{O}_X = \mathcal{O}_Y$. Then every fiber is connected.*
(2) *(Zariski's Main Theorem) Let $f : X \to Y$ be a birational projective morphism of Noetherian varieties, and Y normal. Then every fiber is connected.*

(3) *(Stein factorization) Let* $f : X \to Y$ *be projective,* X, Y *Noetherian. Then* f *factors as a morphism with connected fibers, followed by a finite morphism.*

Proof. (1) If some fiber is disconnected, its H^0 is a nontrivial direct sum. Applying that to the thickenings of the fiber, the second formula is a direct sum of rings. By the theorem, $(f_*\mathcal{O}_X)\hat{}_y$ is a direct sum of rings. By assumption, $(\mathcal{O}_Y)\hat{}_y$ is then a direct sum of rings. But it's local, so it can't be.

(2) It's enough to check locally in Y, so take $Y = \mathrm{Spec}\, A$, and let $B = \Gamma(Y, f_*\mathcal{O}_X)$. Since $f_*\mathcal{O}_X$ is coherent, B is f.g. over A. By the birationality, and since A is normal, $A = B$. So $f_*\mathcal{O}_X = \mathcal{O}_Y$. Now use part (1).

(3) Let the intermediate space be $Y' := \mathbf{Spec}\, f_*\mathcal{O}_X$. Since $f_*\mathcal{O}_X$ is coherent on Y, the map $Y' \to Y$ is finite. It is a little technical to be sure that the map $X \to Y'$ is again projective, but it certainly has $f_*\mathcal{O}_X = \mathcal{O}_{Y'}$.

□

Non-example: Y is a union of two lines, and f is the normalization.

It's a little tricky to combine Zariski & Stein to handle the non-birational case. One wants to say "if the generic fiber is connected", but the fiber over the generic point might be even if a general fiber is not, as in the squaring map $\mathbb{A}^1 \to \mathbb{A}^1$.

19. Equivariant K-theory

Let G be a group, and R a commutative Noetherian ring carrying an action of G by ring automorphisms. Then we can define a G**-equivariant module** M to be one carrying an action of G by abelian-group automorphisms, such that the action map $R \otimes M \to M$ is G-equivariant.

Special case: R is a $\mathbb{C}$-algebra, and $G = T := (\mathbb{C}^\times)^d$. Then an action of T on R is the same as a $\mathbb{Z}^d$-grading on R, and an equivariant module is just a graded module.

Define $K_0^G(R), K^0_G(R)$ using G-equivariant modules. This is already interesting for $R = \mathbb{C}$ carrying the trivial G-action; it is the "representation ring" $\mathrm{Rep}(G)$ of G. Even more specifically, if $G = T = (\mathbb{C}^\times)^d$, then $K_T(\mathrm{Spec}\,\mathbb{C}) \cong \mathbb{Z}[x_1^\pm, \ldots, x_d^\pm]$.

To extend the definition to schemes, we need to be careful about what it means for a group G to act on a scheme X. One possibility is a homomorphism from G to the set of automorphisms of X, i.e. each map $\{g\} \times X \to X$ is a morphism of schemes. But the more interesting possibility is that G is itself a scheme, and the action map $\alpha : G \times X \to X$ (not to be confused with the projection map π) is a morphism of schemes. Then a G**-equivariant sheaf** $\mathcal{F}$ over X is one equipped with an isomorphism $\alpha^*\mathcal{F} \to \pi^*\mathcal{F}$ of sheaves over $G \times X$, satisfying an obvious associativity condition over $G \times G \times X$. With this in place, we can define $K_0^G(X), K^0_G(X)$.

Why care about equivariant K-theory?

1. Representation theory of G is partly reflected in $K^0_G(pt)$, and very accurately so if all short exact sequences of finite-dimensional representations split (G is "reductive"). So it is interesting to see representation theory as the point case of a larger subject.

More specifically, if G is a connected, reductive algebraic group (e.g. a product of simple groups), then one has the Borel-Weil-Bott theorem on sheaf cohomology of $\mathfrak{B}$:

Theorem 22 (Borel-Weil-Bott)**.** *Let $\mathfrak{B}$ denote the scheme of maximal solvable Lie subalgebras of* $\mathrm{Lie}(G)$*, scheme structure from the embedding into a Grassmannian of subspaces of* $\mathrm{Lie}(G)$*. This is well known to be a transitive* G*-scheme hence smooth, and closed hence proper.*

Then for each invertible G*-sheaf* $\mathcal{L}$ *on* $\mathfrak{B}$*, there is at most one* i *such that* $H^i(\mathfrak{B};\mathcal{L}) \neq 0$*, and that* G*-representation is irreducible. (Borel-Weil only treat the* $i = 0$ *case. Bott identifies* i *in terms of* $\mathcal{L}$*.)*

For each $i \leq \dim \mathfrak{B}$*, if we restrict to those* $\mathcal{L}$ *such that* $H^i(\mathfrak{B};\mathcal{L}) \neq 0$*, then we obtain each* G*-irrep exactly once.*

On the one hand, this is more of an advertisement for equivariant sheaf cohomology, rather than K-theory. On the other hand, the fact that all but one sheaf cohomology group vanishes says that the alternating sum defining the holomorphic Euler characteristic doesn't actually lead to any cancelation between different H^i.

2. The *Kirwan injectivity theorem:* if X is smooth projective, and X^T is the subscheme of fixed points, then the restriction map $K^0_T(X) \to K^0_T(X^T) \cong K^0(X^T) \otimes \mathrm{Rep}(T)$ is *injective!* With that, one can prove the following:

3. The *Atiyah-Bott localization theorem.* Assume further that X^T is a finite set. Let $\alpha \in K^0_T(X)$, and $\pi : X \to pt$. Thinking of $\pi_*([\alpha]) \in K^0_T(pt) = \mathrm{Rep}(T)$ as a function on T, we have

$$\pi_*([\alpha]) = \sum_{f \in X^T} \frac{\alpha|_f}{\det(\mathbf{1} - t^{-1}|_{T_f X})}.$$

Applied to the Borel-Weil theorem, this recovers the Weyl character formula.

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