# Mathematical Statistics HW 1

The following solution is provided by Xiao Luo.

0.(a) Note first that $S_X^2 \sim \sigma_X^2\chi_{n-1}^2/(n-1), S_Y^2 \sim \sigma_Y^2\chi_{m-1}^2/(m-1)$.So we can apply Satterthwhaite's method to approximate $(\frac{S_X^2}{n}+\frac{S_Y^2}{m})/(\frac{\sigma_X^2}{n}+\frac{\sigma_Y^2}{m})$ by $c\frac{\chi_\nu^2}{\nu}$ for some constants $c$ and $\nu$. Apply the method of moments by equating the mean and variance of the above two expressions, we can easily find that $c = 1$. And for equating the variance, we have

$$\begin{aligned}
\mathrm{Var}[\frac{\frac{S_X^2}{n}+\frac{S_Y^2}{m}}{\frac{\sigma_X^2}{n}+\frac{\sigma_Y^2}{m}}] &= \frac{\frac{\mathrm{Var}[S_X^2]}{n^2}+\frac{\mathrm{Var}[S_Y^2]}{m^2}}{(\frac{\sigma_X^2}{n}+\frac{\sigma_Y^2}{m})^2} \\
&= \frac{\frac{2(n-1)\sigma_X^4}{n^2(n-1)^2}+\frac{2(m-1)\sigma_Y^4}{m^2(m-1)^2}}{(\frac{\sigma_X^2}{n}+\frac{\sigma_Y^2}{m})^2} \\
&= 2\frac{\frac{\sigma_X^4}{n^2(n-1)}+\frac{\sigma_Y^4}{m^2(m-1)}}{(\frac{\sigma_X^2}{n}+\frac{\sigma_Y^2}{m})^2},
\end{aligned}$$

and

$$\mathrm{Var}[\frac{\chi_\nu^2}{\nu}] = \frac{2}{\nu}.$$

Thus,

$$\nu = \frac{(\frac{\sigma_X^2}{n}+\frac{\sigma_Y^2}{m})^2}{\frac{\sigma_X^4}{n^2(n-1)}+\frac{\sigma_Y^4}{m^2(m-1)}}.$$

Because $S_X^2, S_Y^2$ are unbiased estimators of $\sigma_X^2, \sigma_Y^2$, $\nu$ can be estimated with

$$\hat{\nu} = \frac{(\frac{S_X^2}{n}+\frac{S_Y^2}{m})^2}{\frac{S_X^4}{n^2(n-1)}+\frac{S_Y^4}{m^2(m-1)}}.$$

0.(b)

$$\begin{aligned} T' &= \frac{\bar{X}-\bar{Y}}{\sqrt{\frac{S_X^2}{n}+\frac{S_Y^2}{m}}} \\ &= \frac{(\bar{X}-\bar{Y})/\sqrt{\frac{\sigma_X^2}{n}+\frac{\sigma_Y^2}{m}}}{\sqrt{\frac{S_X^2}{n}+\frac{S_Y^2}{m}}/\sqrt{\frac{\sigma_X^2}{n}+\frac{\sigma_Y^2}{m}}}, \end{aligned}$$

where $\sqrt{\frac{S_X^2}{n}+\frac{S_Y^2}{m}}/\sqrt{\frac{\sigma_X^2}{n}+\frac{\sigma_Y^2}{m}} \sim \sqrt{\chi_\nu^2/\nu}$ and under the null hypothesis, by the central limit theorem, the distribution of $(\bar{X}-\bar{Y})/\sqrt{\frac{\sigma_X^2}{n}+\frac{\sigma_Y^2}{m}}$ can be approximated by a standard normal distribution. Therefore, $T'$ can be approximated by a $t$ distribution with $\hat{\nu}$ degrees of freedom.

1. Because $\underset{\sim}{h}$ is one to one, its inverse is well defined. Then, the likelihood function of $\underset{\sim}{\eta}$ is given by

$$L^*(\underset{\sim}{\eta}|\underset{\sim}{x}) = \prod_{i=1}^{n} f(x_i|\underset{\sim}{h}^{-1}(\underset{\sim}{\eta})) = L(\underset{\sim}{h}^{-1}(\underset{\sim}{\eta})|\underset{\sim}{x}).$$

It follows that

$$\sup_{\underset{\sim}{\eta}} L^*(\underset{\sim}{\eta}|\underset{\sim}{x}) = \sup_{\underset{\sim}{\eta}} L(\underset{\sim}{h}^{-1}(\underset{\sim}{\eta})|\underset{\sim}{x}) = \sup_{\underset{\sim}{\theta}} L(\underset{\sim}{\theta}|\underset{\sim}{x}).$$

Thus, the maximum of $L^*(\underset{\sim}{\eta}|\underset{\sim}{x})$ is attained at $\underset{\sim}{\eta} = \underset{\sim}{h}(\hat{\underset{\sim}{\theta}})$, which implies that the m.l.e. of $\underset{\sim}{\eta}$ is $\underset{\sim}{h}(\hat{\underset{\sim}{\theta}})$.

2. (a) The likelihood function of $\underset{\sim}{\theta}$ is given by

$$L(\underset{\sim}{\theta}|\underset{\sim}{x}) = \frac{1}{\sigma^n}\exp\left[-\frac{1}{\sigma}\sum_{i=1}^{n}(x_i-u)\right] \qquad \text{for} \min_i\{X_i\} \geq u,$$

and otherwise $L(\underset{\sim}{\theta}|\underset{\sim}{x}) = 0$. So we just need to consider the case $u \geq \min_i\{X_i\}$. For any fixed $\sigma > 0$, $L$ is a a decreasing function of $u$ on $[\min_i\{X_i\}, +\infty)$ and is thus maximized at $\hat{u} = \min_i\{X_i\}$. Let $l = \log L$ and by substituting $u$ by $\hat{u}$ and differentiating $l$ with respect to $l$, we can find $l$ is maximized at $\hat{\sigma} = \underset{\sim}{\bar{X}} - \min_i\{X_i\}$ where $\underset{\sim}{\bar{X}}$ is the mean of $\{X_i, i = 1, \cdots, n\}$. Therefore, the m.l.e. of $(u, \sigma)$ is $(\min_i\{X_i\}, \underset{\sim}{\bar{X}} - \min_i\{X_i\})$.

(b)

$$\begin{aligned}
P_{\underset{\sim}{\theta}}[X_1 \geq t] &= \int_t^\infty \frac{1}{\sigma}\exp[-(x-u)/\sigma]\,\mathrm{d}x \\
&= \int_t^\infty \exp[-(x-u)/\sigma]\,\mathrm{d}(\frac{x-u}{\sigma}) \\
&= \int_{\frac{t-u}{\sigma}}^\infty \exp[-y]\,\mathrm{d}y \\
&= \exp[(u-t)/\sigma].
\end{aligned}$$

So $P_{\underset{\sim}{\theta}}[X_1 \geq t]$ is a function of $\underset{\sim}{\theta}$ and by the invariance property of m.l.e., the m.l.e. of $P_{\underset{\sim}{\theta}}[X_1 \geq t]$ is $\exp[(\min_i\{X_i\} - t)/(\bar{\underset{\sim}{X}} - \min_i\{X_i\})]$.

3. The likelihood function of $\theta$ is given by

$$L(\theta|\underset{\sim}{x}) = 1, \quad \text{if } X_{(n)} - \frac{1}{2} \leq \theta \leq X_{(1)} + \frac{1}{2},$$

and otherwise $L(\theta|\underset{\sim}{x}) = 0$. So $\sup_\theta L(\theta|\underset{\sim}{x}) = 1$ and can thus be attained at any $\theta$ satisfying $X_{(n)} - \frac{1}{2} \leq \theta \leq X_{(1)} + \frac{1}{2}$. In other words, any $T$ such that $X_{(n)} - \frac{1}{2} \leq T \leq X_{(1)} + \frac{1}{2}$ is a m.l.e. of $\theta$.

4. The likelihood function of $\underset{\sim}{\theta} = (\mu, \sigma^2)$ is given by

$$f(\underset{\sim}{x}, \mu, \sigma^2) = \prod_{i=1}^n [\frac{9}{10\sigma}\varphi(\frac{x_i - \mu}{\sigma}) + \frac{1}{10}\varphi(x_i - \mu)].$$

Note that for any $j = 1, \cdots, n$,

$$\begin{aligned}
f(\underset{\sim}{x}, x_j, \sigma^2) &= \prod_{i=1}^n [\frac{9}{10\sigma}\varphi(\frac{x_i - x_j}{\sigma}) + \frac{1}{10}\varphi(x_i - x_j)] \\
&\geq \frac{9}{10\sigma}\varphi(0) \times \prod_{1\leq i\leq n, i\neq j} \frac{1}{10}\varphi(x_i - x_j),
\end{aligned}$$

hence by letting $\sigma \to \infty$, we have $\sup_{\mu,\sigma^2} f(\underset{\sim}{x}, \mu, \sigma^2) = \infty$ and hence no m.l.e. estimator exists for $\underset{\sim}{\theta}$. By the above arguments, we know $\sup_{\mu,\sigma} f(\underset{\sim}{x}, \mu, \sigma^2) = \infty$, and $\sup_\sigma f(\underset{\sim}{x}, \hat{\mu}, \sigma^2) = \infty$ if $\hat{\mu}$ equals one of the numbers $x_1, \cdots, x_n$, so we just need to show $\sup_{\mu,\sigma} f(\underset{\sim}{x}, \mu, \sigma^2)$ is finite if $\mu$ is not equal to any $x_i$. And it is equivalent to showing for any $i$, $\frac{9}{10\sigma}\varphi(\frac{x_i-\mu}{\sigma}) + \frac{1}{10}\varphi(x_i - \mu)$ is finite. Since $\varphi$ is the standard normal density function, we know $\frac{1}{10}\varphi(x_i - \mu)$ is bounded and also by analytical method $\frac{9}{10\sigma}\varphi(\frac{x_i-\mu}{\sigma})$ is also bounded for $\mu \neq x_i$.