Problem 0

The assumption of equal variances, which was made in Exercise 8.41, is not always tenable. In such a case, the distribution of the statistic is no longer a t. Indeed, there is doubt as to the wisdom of calculating a pooled variance estimate. (This problem, of making inference on means when variances are unequal, is, in general, quite a difficult one. It is known as the Behrens–Fisher Problem.) A natural test to try is the following modification of the two-sample t test: Test

\[ H_0: \mu_X = \mu_Y \quad \text{versus} \quad H_1: \mu_X \neq \mu_Y, \]

The exact distribution of \( T' \) is not pleasant, but we can approximate the distribution using Satterthwaite's approximation (Example 7.2.3).

(a) Show that

\[ \frac{s_X^2}{n} + \frac{s_Y^2}{m} \sim \chi^2 \quad \text{(approximately)}, \]

where \( \nu \) can be estimated with

\[ \nu = \frac{\left( \frac{s_X^2}{n} + \frac{s_Y^2}{m} \right)^2}{\frac{s_X^4}{n(n-1)} + \frac{s_Y^4}{m(m-1)}}. \]

(b) Argue that the distribution of \( T' \) can be approximated by a \( t \) distribution with \( \nu \) degrees of freedom.

Problem 1. Let \( \mathbf{X} \) be a random vector distributed according to the probability measure \( P^{\theta} \). Suppose that \( \hat{\theta} \) is the M.L.E. of \( \theta \). Define \( U = h(\theta) \) and let \( f_{U}(X) \) denotes the density function of \( X \) in terms of \( U \). Show that the M.L.E. of \( U \) is \( h(\theta) \). (Hence M.L.E. is unaffected by reparametrization.) Assume that \( h \) is a one to one function.

Problem 2. Let \( X_1, \ldots, X_n \) be i.i.d. with density of \( X_1 \) being

\[ f(x, \theta) = \frac{1}{\sigma} \exp\left[-(x-u)/\sigma\right] \quad x \geq u \]

where \( \theta = (u, \sigma^2) \), \( u, \sigma^2 \) both unknown \( u \in \mathbb{R} \) and \( \sigma > 0 \).

(a) Find the m.l.e. of \( u \) and \( \sigma^2 \)

(b) Find the m.l.e. of \( P^{\theta}[X_1 > t] \) for \( t > u \).

Problem 3. Let \( X_1, \ldots, X_n \) be a sample (i.e.i.i.d) from a uniform distribution over \([\theta - \frac{1}{2}, \theta + \frac{1}{2}]\). (Note: The distribution has positive density at the end points \( \theta - \frac{1}{2} \) and \( \theta + \frac{1}{2} \)). Show that any \( T \) such that

\[ X(n) - \frac{1}{2} \leq T \leq X(1) + \frac{1}{2} \]

is a m.l.e. of \( \theta \), where \( X_{(1)} = \min X_i \) and \( X_{(n)} = \max X_i \).