

Problem 0

Math 674 HW 1, Spring 2006

The assumption of equal variances, which was made in Exercise 8.41, is not always tenable. In such a case, the distribution of the statistic is no longer a t . Indeed, there is doubt as to the wisdom of calculating a pooled variance estimate. (This problem, of making inference on means when variances are unequal, is, in general, quite a difficult one. It is known as the *Behrens-Fisher Problem*.) A natural test to try is the following modification of the two-sample t test: Test

$$H_0: \mu_X = \mu_Y \quad \text{versus} \quad H_1: \mu_X \neq \mu_Y,$$

$\nu = 1$

The exact distribution of T' is not pleasant, but we can approximate the distribution using Satterthwaite's approximation (Example 7.2.3).

(a) Show that

$$\frac{\frac{s_X^2}{n} + \frac{s_Y^2}{m}}{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}} \sim \frac{\chi_\nu^2}{\nu} \quad (\text{approximately}),$$

where ν can be estimated with

$$\hat{\nu} = \frac{\left(\frac{s_X^2}{n} + \frac{s_Y^2}{m} \right)^2}{\frac{s_X^4}{n^2(n-1)} + \frac{s_Y^4}{m^2(m-1)}}.$$

(b) Argue that the distribution of T' can be approximated by a t distribution with $\hat{\nu}$ degrees of freedom.

Problem 1. Let \underline{X} be a random vector distributed according to the probability measure $P_{\underline{\theta}}$. Suppose that $\hat{\underline{\theta}}$ is the M.L.E. of $\underline{\theta}$. Define $\underline{\eta} = \underline{h}(\underline{\theta})$ and let $f_{\underline{\eta}}(\underline{x})$ denotes the density function of \underline{X} in terms of $\underline{\eta}$. Show that the M.L.E. of $\underline{\eta}$ is $\underline{h}(\hat{\underline{\theta}})$. (Hence M.L.E. is unaffected by reparametrization.) Assume that h is a one to one function.

Problem 2. Let X_1, \dots, X_n be i.i.d. with density of X_1 being

$$f(x, \underline{\theta}) = \frac{1}{\sigma} \exp[-(x-u)/\sigma] \quad x \geq u$$

where $\underline{\theta} = (u, \sigma^2)$, u, σ^2 both unknown $u \in \mathbb{R}$ and $\sigma > 0$.

(a) Find the m.l.e. of u and σ^2

(b) Find the m.l.e. of $P_{\underline{\theta}}[X_1 \geq t]$ for $t > u$.

Problem 3. Let X_1, \dots, X_n be a sample (i.e. i.i.d) from a uniform distribution over $[\theta - \frac{1}{2}, \theta + \frac{1}{2}]$. (Note: The distribution has positive density at the end points $\theta - \frac{1}{2}$ and $\theta + \frac{1}{2}$). Show that any T such that

$$X_{(n)} - \frac{1}{2} \leq T \leq X_{(1)} + \frac{1}{2} \quad \text{is a m.l.e. of } \theta,$$

where $X_{(1)} = \min X_i$ and $X_{(n)} = \max X_i$.