Asking good questions in the mathematics classroom[1]

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I want to thank the organizers for inviting me to share a personal case study in introducing reform; reform in how I teach mathematics and what mathematics I teach. The context in this case study is calculusÑbut the process of change, experimentation, and discovery I describe are not tied specifically to calculus.

In what ways have I changed how I teach? What motivated me to seek a change? What changes have I observed in myself as a result? What changes have I observed in my students, and in my colleagues?

IÕll start with a bit of history--I have been teaching mathematics off and on for over 30 years. I started as a student teacher at Maloney High School in Meriden, Connecticut in 1972. Since that time IÕve taught at a small liberal arts college for women, two state universities, and IÕm currently in the Mathematics Department at Cornell. For most of those years I taught in what I would call a traditional way. You might wonder why it took me almost 30 years to initiate a serious change in my approach to teaching.

Actually for 10 of the past 13 years, I was in academic administration full-time. Technically I had left academics, but I taught one course a year as an adjunct in an attempt to keep in touch with the real purpose of the university and to try not to lose my soul. My teaching was in a very comfortable groove so I never really thought about it. A little over two years ago, I left administration and resumed a full-time academic position in the Mathematics DepartmentÑI took on the responsibility of training and supervising more than 80 teaching assistants who TA for our department each year. Suddenly, I was the ÒexpertÓ on how to teach. I was responsible for helping our eager, young, prospective faculty develop their skills as teachers. I hit the books. I had of course heard over the years about the great revolution and reform movements that were taking place while I was in exile in administration. I had been an interested observer. Now the more I read

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and heard about efforts to improve student learning, the more I knew I wanted to experiment with some of these different pedagogical approaches myself, and to share what I might learn with our graduate students.

I started with some noble goals that I distilled from the collective wisdom found in recent publications[2], and that are undoubtedly familiar to you:

Goals:

- To foster an active learning environment
- To stimulate studentsÕ interest and curiosity in mathematics;
- To offer students frequent opportunities to make conjectures and argue about their validity;
- To help students monitor their understanding;

But soon I wondered, ÒLearn what? ÓÒUnderstand what?ÓÒHow is it that students learn mathematics?ÓÒWhat will they have to discuss, let alone argue about?ÓSo I hit the books again, mostly articles and monographs on calculus reform. That seemed to be where quite a bit of experimentation and research had been done. What did we know about how college students learn mathematics in general, and calculus in particular? As I read what struck me most strongly was that

- Students come to class with both prior knowledge and some misconceptions about some of the key concepts in calculus. How could I identify that prior knowledge and those misconceptions? How could I create experiences for students to connect, compare, and revise what they already knew, to incorporate the new ideas in calculus?
- I needed a method that would help me monitor —frequently-- what my students already knew, and what they were learning. I wanted a method whereby I could promptly redirect my efforts so as to build on their previous knowledge, intuition, and understanding.

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[2] National Research Council (2002) *Learning and Understanding: Improving advanced study of Mathematics and Science in U.S. High Schools*, Committee on Programs for Advanced Study of Mathematics and Science in American High Schools. Jerry P Gollub, Meryl W. Bertenthal, Jay Labov, Philip C. Curtis, eds.. Center for Education Division of Behavioral and Social Sciences and Education. Washington, DC: National Academy Press

It struck me that the ideal method would resemble a Socratic dialogue. Not exactly a Ònew ideaÓ But perhaps that was it. I should just ask my students questions; good questions that would help us engage in a dialogue.

The Method

About five years earlier, I had met Shelia Tobias when she visited a mutual friend at Cornell. Shelia told me about a fabulously successful approach a physicist named Eric Mazur had developed for teaching introductory physics at Harvard. He called it ConcepTests and Peer Instruction and she had showcased it in one of her books[3]. Shelia was quite convinced there was something significant happening in MazurÕs classroom. She gave me a copy of her book. I put in on my bookshelf. There it was on my shelf, for almost six years. Now I read it. It was a powerful approach. At the heart of the method were good questions, powerful questions that stimulated discussion and debateÑabout physics. Here is an example[4]:

> A boat carrying a large boulder is floating on a lake. The boulder is thrown overboard and sinks. The water in the lake (with respect to the shore)
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> 1. rises.
> 2. drops.
> 3. remains the same.

What a great way to launch a discussion and test studentsÕunderstanding of fluid statics. One could imagine the experience. The question was non-computational and purely conceptual. The answers are all attractive and each represented at least a partial understanding that could reasonably be argued. And since the correct answer was somewhat surprising (the water in the lake with respect to the shore drops), the question would undoubtedly provide a hook for remembering a key concept in physics.

I was excited. I wondered if it would be possible to craft such questions in calculus; questions that were non-computational, that were related to students experiences, questions that were memorable, and surprising, that helped build on students partial understanding. I wondered if instructors would take time out from class to use good questions if they were attractive, if they led to active lively discussions, and if they helped students connect their intuitive understanding of the world to the concepts of

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[3] Tobias, Shelia. (1992) *Revitalizing Undergraduate Science: Why some things Work and Most DonÕt,* Tuscon, AZ: Research Corporation.

[4] Mazur, Eric. (1997) *Peer Instruction: A UserÕs Manual,* Upper Saddle River, New Jersey: Prentice Hall

calculus. A group of us at Cornell[5] resolved to try our hand at writing some good questions about calculus, and with the support of a Òproof of conceptÓ grant from NSF, weÕve been doing just that this term. The questions span a range of concepts and applications. Here is my very first questionÑIt came to me on my way to Mathfest, as I was sitting in the airport, reading a Wall Street Journal IÕd found on a seat:

> An article in the Wall Street JournalÕs ÒHeard on the StreetÓ column (*Money and Investment* August 1, 2001) reported that investors often look at the Òchange in the rate of changeÓ to help them Òget into the market before any big rallies.Ó Your stock broker alerts you that the rate of change in a stockÕs price is increasing. As a result
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> a) you can conclude the stocks price is increasing
> b) you cannot determine whether the stockÕs price is increasing or decreasing
> c) you can conclude the stockÕs price is decreasing

Now this may not seem earth shaking, but I tried it out this term, and my students really liked it. They had some very good discussions, quite a bit related to unpacking the language. In the end, Òchange in the rate of changeÓ no longer sounded double talk. It sounded like calculus.

So thatÕs our approach, ask the students good questions. ThatÕs the process by which we hope to engage students in thinking and talking about mathematics, right there in class. ThatÕs the process that we find helps us, as instructors, think more deeply about what we are teaching and how to communicate it. Writing good questions as it turns out is very hard work. IÕll share some of them with you at the breakout session, and hope you can make suggestions for improvements-and share ideas of your own.

During the planning phase of this project I thought that 90% of our efforts would be devoted to the pedagogy. After all, calculus is pretty standard. I expected very little time or effort would be devoted to rethinking the content. Well, I was wrong.

The Content

We started by wanting to raise the visibility of key concepts. I thought I knew what those were. I was focusing mainly on applications; ways of connecting calculus to studentsÕ experiences, real or imagined. Continuing along with that in mind I wrote more questions. My second question was inspired by the behavior of our eldest son who at 18 had just taken up flying and was eager to try it without the plane:

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[5] The ideas discussed here have been influenced by discussions and ongoing work with Robert Connelly, Oscar Rothaus, Bob Strichartz, David Henderson and a group of graduate students in mathematics who are working on an NSF funded curriculum development project to create good questions in calculus.

Imagine that you are sky-diving. The graph of your speed as a function of time, from the time you jumped out of the plane to the time you achieve terminal velocity is most likely

a) Increasing concave down.
b) Decreasing concave down.
c) A straight line with positive slope.
d) Increasing concave up.

What could be better than jumping out of an airplane? I could imagine students feeling the wind rushing by their faces, at a speed that was increasing, but at a slower rate. This question would be fun to discuss, and I can tell you it is. We were on a roll. But there was something missing.

One of the things that is so powerful about MazurÕs ConcepTests is that they engage the students in thinking about the physics. Was there a way to craft questions that would get students talking about the mathematics? What are the key mathematical ideas weÕd like them to talk about? What are the fundamental ideas that make calculus work? We decided to look at the theorems. How could we embed them into some meaningful contexts? How could we connect them to studentsÕintuition? How could we make those discussions memorable? Here is an example:

> True or False: (Be ready to offer a proof or counter example): You were once exactly three feet tall.

I tried it out in both of my calculus classes this term, and I can tell you this question really knocks students for a loop. Of course I was once three feet tall. But what did that have to do with calculus? How could they convince their neighbor? We had ideas from getting old baby pictures to using the height marks on the kitchen door frame. Ultimately students argued that newborn babies are usually about eighteen inches, and they had managed to grow to their current heightÑwell over three feet. But how could they prove they were once **exactly** three feet tall? Did they need to know when that occurred? Or simply that it had? It is interesting that a few students raised the point that perhaps they were never exactly three feet tall. Perhaps they just added a molecule and skipped over the 36 inch mark. We found ourselves discussing not only the application of the Intermediate Value Theorem, but the wisdom of assuming that growth was a continuous function of time.

I found this question worked even better in my second section when I followed it with:

> At the start of second quarter, your basketball team had 18 points, by half-time they had 48. True or False (Be ready to give a proof or a counter example) There was at least one moment in the first half when your team had exactly 36 points.

Well of course thatÕs not necessarily true, but interestingly to me, some students argued that it was true because it MIGHT be true. So what did my students and I learn?

1) We debated about continuity. We talked about why these two similar questions had such radically different answers.
2) We discussed the idea that often we choose a model, in the case of continuity itÕs rarely an exact model, but rather one that simplifies the representation.
3) We discussed logic and mathematical reasoning: how one uses a theorem, the need to verify that the hypotheses are satisfied, and the nature of a counterexample. If a statement is true sometimes, but not necessarily always, can we say itÕs true?

The whole discussion took 10-15 minutes. It was fun, I could see they were thinking about the issue of continuity. Frankly, I wasnÕt surprised when these students did better than my students had in previous years, on the Intermediate Value Theorem question on the exam. But more importantly I felt they had glimpse of the spirit of what makes mathematics enjoyable. It is a subject of ideas not just algorithms, of understanding not just applications.

In an effort to raise the visibility of continuity further, I decided to try the next question:

YouÕve decided to estimate e^2 by squaring progressively longer decimal approximations of e.

a) This is a good idea because e is rational.
b) This is a good idea because $y=x^2$ is a continuous function.
c) This is a good idea because $y=e^x$ is a continuous function.
d) This is a bad idea because e is irrational.

The Surprise

What would students do with such a question? Would it be possible for them to connect their strong, natural, intuition about how to approximate e^2 to the definition of continuity? Would they find the problem too difficult?, Too abstract?

To my great surprise I found my students really liked this question. It raised the issue of what continuity is from a practical point of view. The incorrect answer c) helped clarify what continuity meant, by seeing that we were not approaching e^2 by sneaking up on 2.

What was entirely unexpected though, was that this question ignited a string of studentsÕ questions not about calculus, but ÒsimplyÓ about numbers. What are irrational numbers? How DO we multiply or even add two infinite non repeating decimals? And if you always need to chop off part of the number in order to work with it, how could you tell if two numbers were equal or not? For example are $.\overline{9}$ and 1 the same number? If we subtract one from the other what do we get? Once we opened up the topic for discussion the questions flooded in. Not only was I asking them questions, they were asking me. It was a challenge to answer them. Simple questions, complex ideas. Should I punt and tell them these are the type of things we discuss in a 400 level analysis course that 99% of them would never take? No, my students were intrigued with infinite decimals. TheyÕd learned how to add, subtract, multiply, and divide rational numbers, but no one had ever raised the issue of how one might compute with irrationals. It suddenly became clear, if we were going to discuss questions like these, we would need to know more about that bedrock of calculusÑthe real numbers.

Long ago I gave up on the notion of discussing the least upper bound principle in first semester calculus, and frankly I still have. But until this term, IÕd never thought seriously about introducing completeness through a discussion of decimal representations of real numbers. Decimals are familiar. Students know that infinite non-repeating decimal numbers exist. TheyÕve seen the proof that $\sqrt{2}$ is irrational. But theyÕve never taken the next step and tried to operate with these numbers. So here are my latest questions and they are for us:

Could it be that if we discuss real numbers by trapping them between finite decimals that we can start laying the groundwork for how we define so many concepts in calculus, by squeezing in on them? By not talking in an accessible way about real numbers, are we discouraging students from developing their intuition about them? Is there a truly accessible way to talk about the real numbers? Can we carve out time for questions and discussions about numbers and still get through the material? How can we expect students to deepen their understanding of what calculus is about, if theyÕve never been encouraged to explore the nature of the real numbers? How can we encourage them to ask not only how to use calculus, but how calculus works? If students spend more time talking about ideas and we spend less time working problems and giving detailed explanations, what will they learn? Will their computational skills suffer? Will they be ready for the next course? Will they like mathematics more? Will they like it less?

I will say in closing that I have begun to think quite differently about what the key ideas are in calculus. I have been surprised to learn what kinds of questions students find interesting. I would like to end with the questions that the organizers posed in describing this workshop:

What do teachers need to know? What kinds of experiences should they have in their own mathematics courses?

I look forward to some stimulating discussions during the breakout session.