Liquidity Risk and Trade Impact
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The CJP model - Cetin, Jarrow, Protter (2004)

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Self-financing strategies (X, Y) satisfy

$$Y_T = Y_0 + \int_0^T X_u - dS_u - \int_0^T M_u d[X]_u.$$

$X_t$ denotes the number of shares held at time $t$ and $Y_t$ the money in the bank account (0 interest).
Figure: Typical order book density for linear model.
Figure: Order book is partly used up.
Figure: Price Impact at time $t + \cdot$
Mathematical Framework

We take the point of view of an “impatient” investor. All the trades are made at the market price $S^0(t,x)$. 

Second type of resiliency: damping. In the long run, the effect of past trades on prices decreases: price $S^0_t$ converges to the unaffected price $S_t$. 
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- **Second type of resiliency : damping.** In the long run, the effect of past trades on prices decreases: price $S^0_t$ converges to the unaffected price $S_t$. 
Figure: Typical sample path
Mathematical Framework

- We define the price after impact by

\[ S_{t+}^0 = S_t + 2\lambda \int_0^t e^{-\kappa(t-u)} M_u dX_u + 2\lambda \int_0^t e^{-\kappa(t-u)} d[M, X]_u. \]

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- Let \( \sigma_n : 0 = \tau_0^n \leq \tau_1^n \leq \ldots \leq \tau_{k_n}^n = t \) be a sequence of random partitions tending to the identity and \( \Delta_k^n X = X_{\tau_k^n} - X_{\tau_{k-1}^n} \). A pair \( (X_t, Y_t)_{t \geq 0} \) is a self-financing trading strategy (s.f.t.s) if \( X \) is a cadlag process and \( Y \) is an optional process satisfying

\[ Y_t = Y_0 - \lim_{n \to \infty} \sum_{k=1}^{k_n} \Delta_k^n X S_0^0(\tau_k^n, \Delta_k^n X). \]

We will always define trading strategies with \( X_{0-} = Y_{0-} = 0 \).
Self-financing Strategies

**Theorem 1**

Let $X$ be a FV cadlag process and $Y$ an optional process. Define

$$
I_t = \lambda \int_0^t X_u^2 d m_u
$$

$$
L_t = \int_0^t K(t-u) M_u d [X,X]_u + (1-\lambda) M_t X_t^2
$$

with $K(t) = 1 - \lambda e^{-\kappa t}$ and $m_t = e^{-\kappa t} M_t$. If $(X_t, Y_t)_{t \geq 0}$ is a self-financing trading strategy then

$$
Y_T + X_T S_T^0 (\mathbf{-X}_T) = Y_0 + X_0 S_0(X_0) + \int_0^T X_u - d S_u - L_T - I_T
$$

- The case $\lambda = 0$ corresponds to the CJP model. (Full resiliency)
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No Arbitrage and Equivalent Martingale Measures

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**Theorem 2**

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- Profit $= \int_0^T X_u dS_u - \lambda \int_0^t X_u^2 dm_u -$ Liquidity Costs.
The Model

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\[ dv_t = \tilde{a}(v_s, s)ds + b_1(v_s, s)dW_{1,s} + b_2(v_s, s)dW_{2,s} \]
\[ dm_t = \eta(m_s, s)ds + \xi_1(m_s, s)dW_{1,s} + \xi_2(m_s, s)dW_{2,s} \]

for all \( 0 \leq t \leq T \) in which \( \tilde{a}, b, \xi, \eta \ldots \) are Lipschitz functions which ensures the existence of such processes. \( m_t = \exp(-\kappa(T-t))M_t \). \( v_t \) is a component of the stochastic volatility.
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\( v_t \) is a component of the stochastic volatility.
- Furthermore we will assume that the matrix

\[ \Sigma(m_t, v_t, t) = \begin{pmatrix} b_1(v_t, t) & b_2(v_t, t) \\ \xi_1(m_t, t) & \xi_2(m_t, t) \end{pmatrix} \]

is invertible for all \( 0 \leq t \leq T \).
Recall that \( S_t \) is the stock price resulting in the actions of all investors in the market except me. The aggregated effect of trading done by investor \( i \) on the stock price is \( 2\lambda \int_0^t m_u dX^i_u + 2\lambda [m, X^i]_t \).
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We can expect the stock price to be affected by sum over all investors in the market (except me) and some other risk source: $S_T = S_0 + \sum_i 2\lambda \int_0^T m_u dX_u^i + 2\lambda \sum_i [m, X^i]_T + \int_0^T \nu_u S_u dW_u$. 
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\]

If we assume \( \sum_i dX_t \) is of the form \( S_t dW_t \), the stock process is then given by
\[
dS_t = \mu_t S_t dt + \sum_{i=1}^3 \sigma_i \sigma_t S_t dW_{i,t}
\]
for all \( 0 \leq t \leq T \) in which \( \sigma_t = m_t + \nu_t \).
The Volatility Swaps

We add two volatility swaps, denoted $G_{i,t}$ for $i = 1, 2$. To ensure no arbitrage, we assume the existence of an equivalent probability measure $Q$ such that $S$ is martingale, $m$ is submartingale and

$$G_{i,t} = E_Q(\sigma_{T_i} \mid \mathcal{F}_t)$$

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- S.f.t.s now satisfy

$$Y_T = Y_0 + \int_0^T X_u dS_u - \lambda \int_0^T X_u^2 dm_u - \int_0^T K(T-u)M_u d[X]_u$$

$$+ \sum_i \int_0^T \chi_{i,u} dG_{i,u} - \sum_i \int_0^T K_i(T-u)N_i d[\chi_i]_u.$$

Here $N_i$ and $K_i$ refer to the liquidity constraints of $G_i$. 

Girsanov’s Theorem

By Girsanov’s theorem, there exists a predictable process $\theta$ such that under $Q$

$$B_t = W_t + \int_0^t \theta_s ds.$$ 

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This essentially means that

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(\theta_1, t\sigma_1 + \theta_2, t\sigma_2 + \theta_3, t\sigma_3)\sigma_t = \mu_t \quad \text{and} \quad (\theta_1, t\xi_1 + \theta_2, t\xi_2) \leq \eta(m_t, t).
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\]

- Let \( \zeta_t = \eta(m_t,t) - (\theta_1,t\xi_1(m_t,t) + \theta_2,t\xi_2(m_t,t)) \) and \( a_t = \tilde{a}(v_t,t) - (\theta_1,t b_1(v_t,t) + \theta_2,t b_2(v_t,t)) \). Then

\[
\begin{align*}
dv_t &= a_t dt + b_1(v_t,t)dB_{1,t} + b_2(v_t,t)dB_{2,t} \\
dm_t &= \zeta_t dt + \xi_1(m_t,t)dB_{1,t} + \xi_2(m_t,t)dB_{2,t} \\
dS_t &= \sum_i \sigma_i \sigma_t S_t dB_{i,t}.
\end{align*}
\]
Approximate Completeness and S.f.t.s.

Recall the definition of self-financing:

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Y_T = Y_0 + \int_0^T X_u - dS_u - \lambda \int_0^T X_u^2 - dm_u - \int_0^T K(T - u) M_u d[X]_u \\
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Lemma 1

Fix \( t \) and let \( H_T \in \mathcal{L}^\infty \). Suppose there exist predictable processes \( X \) and \( \chi \) with \( H_T = c + \int_t^T X_u dS_u + \sum_{i=1,2} \int_t^T \chi_{i,u} dG_{i,u} - \lambda \int_t^T X_u^2 dm_u \) for some \( c \in \mathbb{R} \). Then there exists a sequence of s.f.t.s. \( (X^n, \chi^n, Y^n) \) with \( X^n \) bounded, continuous and of finite variation such that \( X^n_t = 0 \), \( X^n_T = 0 \), \( \chi^n_t = 0 \), \( \chi^n_T = 0 \) and \( Y^n_T = \mathbb{E}_Q \left( H_T + \lambda \int_t^T (X^n_{u-})^2 m_u \zeta_u du \bigg| \mathcal{F}_t \right) \) for all \( n \) and \( Y^n_T \rightarrow H_T \) in \( \mathcal{L}^2(dQ) \)
Quadratic Growth BSDEs

- For a given payoff $H_T$, the replication problem boils down to finding processes $X, \chi$ and a constant $c$ that satisfy:

$$H_T = c + \int_t^T X_u dS_u + \sum_{i=1,2} \int_t^T \chi_{i,u} dG_{i,u} - \lambda \int_t^T X_u^2 dm_u.$$
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- The existence of such a process is given by the existence of a solution to quadratic growth BSDEs:

\[ X_t = H - \int_t^T X_s dS_s + \lambda \int_t^T X_s^2 dm_s - \sum_i \int_t^T \chi_{i,s} dG_{i,s} \quad (1) \]

\textbf{Theorem 3}

Let $0 \leq t_0 \leq T$ and $T_1 \neq T_2$. Suppose $\zeta_t = \zeta m_t$ and $a_t = av_t$ for some constants $\zeta \neq a$. For $H \in \mathcal{L}^\infty(\mathcal{F}_T)$, there exists a unique solution $(X_t, \chi_t, Y_t)_{t_0 \leq t \leq T}$ to the following BSDE

$$Y_t = H - \int_t^T X_s dS_s + \lambda \int_t^T X_s^2 dm_s - \sum_i \int_t^T \chi_{i,s} dG_{i,s} \quad (1)$$

for $t_0 \leq t \leq T$. 
The claim $H$ has liquidity constraints associated to it: the replicating cost of $\epsilon H$ is not $\epsilon$ times the replicating cost of $H$. 

Let $(X_{\epsilon}, \chi_{\epsilon}, Y_{\epsilon})$ be the solution of BSDE with terminal condition $\epsilon H$. Let $H_t(\epsilon)$ be the replicating price per share of $\epsilon$ shares of the claim $H$ starting at time $t$, i.e. $Y_{\epsilon t \epsilon}$. Denote $H_t(0) = \lim_{\epsilon \to 0} H_t(\epsilon) = \lim_{\epsilon \to 0} Y_{\epsilon t \epsilon}$ and $H_t'(0)$ the derivative at zero (we will see that it exists).
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Denote $H_t(0) = \lim_{\epsilon \to 0} H_t(\epsilon) = \lim_{\epsilon \to 0} \frac{Y^\epsilon_{t \epsilon}}{\epsilon}$ and $H'_t(0)$ the derivative at zero (we will see that it exists).
The BSDE for $\epsilon H$ can be written in the form:

$$Y_t^\epsilon = \epsilon H + \int_t^T \zeta m_s \lambda(X_s^\epsilon)^2 ds + \sum_i \int_t^T \left( - X_s^\epsilon \sigma_i \sigma_s + \lambda(X_s^\epsilon)^2 \zeta(m_s, s) - \sum_j \chi_j^\epsilon e^{-a(s-T_j)} b_j(v_s, s) \right) dB_i, s$$
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$$\left. - \sum_j \chi_j^{\epsilon, s} e^{-a(s-T_j)} b_j(v_s, s)\right) dB_{i,s}$$

**Theorem 4**

Let $H \in \mathcal{L}^\infty(\mathcal{F}_T)$. Then, if $(X, \chi, Y)$ denotes the solution of the BSDE with $\zeta = 0$ and $\epsilon = 1$, we have that $H_t(0) = Y_t = \mathbb{E}_Q(H|\mathcal{F}_t)$ and

$$\frac{1}{\epsilon}X^\epsilon \to X \text{ in } \mathcal{L}^2(dQ \times dt) \text{ as } \epsilon \to 0.$$
In the results above, $H$ cannot depend on $X$. We don’t know the existence of a solution to the equation

$$\epsilon h(S_T - 2\lambda \int_0^T \tilde{X}_s^\epsilon dm_s) = \tilde{Y}_t^\epsilon - \int_t^T \tilde{X}_s^\epsilon dS_s + \lambda \int_t^T (\tilde{X}_s^\epsilon)^2 dm_s - \sum_i \int_t^T \tilde{\chi}_i, s dG_{i,s}.$$
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Instead, we find solution $(X^\epsilon, \chi^\epsilon, Y^\epsilon)$ of the BSDE

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$$

**Theorem 5**

If $h$ is Lipschitz continuous and bounded then

$$
\sqrt{E_Q \left| \epsilon h(S_T - 2\lambda \int_0^T X_s^\epsilon dm_s) - \epsilon h(S_T - 2\lambda \int_0^T \epsilon X_s dm_s) \right|^2} = O(\epsilon^{2.5}).
$$

Furthermore, if $h$ is twice differentiable and its second derivative is bounded, then

$$
H'_t(0) = E_Q \left( \int_t^T \zeta m_s X_s^2 ds \mid F_t \right) - 2\lambda E_Q \left( h'(S_T) \left( \int_t^T X_s dm_s \right) \mid F_t \right).
$$
References


