Suppose that $G = H \oplus K$ with $G$ a finite abelian group.

(a) How does one get the invariant factors of $G$ directly from those of $H$ and $K$?

Write all the invariant factors of $H$ and $K$ in a list. Take the lcm of the factors in this list. This is the first invariant factor of $G$. For each prime power in the lcm, find an element in the list with the same prime power and divide this element by the prime power. Then we have a new list. Take the lcm of this list, and this is the second invariant factor of $G$. Again, continue this process to generate a new list. Repeat this process until all elements in the list are 1.

**Example.** Let $Inv(H) = \{7 \cdot 5^2 \cdot 3 \cdot 2, 5^2 \cdot 3\}$ and $Inv(K) = \{13 \cdot 3^3 \cdot 2^2, 3 \cdot 2^2, 2\}$.

$List_1 : \{7 \cdot 5^2 \cdot 3 \cdot 2, 5^2 \cdot 3, 13 \cdot 3^3 \cdot 2^2, 3 \cdot 2^2, 2\}$

$lcm_1 : 13 \cdot 7 \cdot 5^2 \cdot 3^3 \cdot 2^2$

$List_2 : \{3 \cdot 2, 5^2 \cdot 3, 1, 3 \cdot 2^2, 2\}$

$lcm_2 : 5^2 \cdot 3 \cdot 2^2$

$List_3 : \{2, 3, 1, 3, 2\}$

$lcm_3 : 3 \cdot 2$

$List_4 : \{1, 1, 1, 3, 2\}$

$lcm_4 : 3 \cdot 2$

$List_5 : \{1, 1, 1, 1\}$ So now we’re done.

$Inv(G) = \{13 \cdot 7 \cdot 5^2 \cdot 3^3 \cdot 2^2, 5^2 \cdot 3 \cdot 2^2, 3 \cdot 2, 3 \cdot 2\}$

(b) Given the invariant factors of $G$ and $H$, how does one calculate the invariant factors of $K$?

Decompose the invariant factors of $G$ and $H$ into their prime powers. If a prime power appears in both lists, delete that prime power from the lists. After all common prime powers have been deleted, the list for $H$ will be empty. Then with the new list for $G$, find the greatest prime power for each prime and multiply them together to find the first invariant factor of $K$. Delete these prime powers from the list. Then apply this process with the new list to find the second invariant factor of $K$. Repeat this process until the list is empty.
Example. Let $\text{Inv}(G) = \{13 \cdot 7 \cdot 5^2 \cdot 3^3 \cdot 2^2, 5^2 \cdot 3 \cdot 2^2, 3 \cdot 2, 3 \cdot 2\}$ and $\text{Inv}(H) = \{7 \cdot 5^2 \cdot 3 \cdot 2, 5^2 \cdot 3\}$.

$\text{List}_H : \{7, 5^2, 5^2, 3, 3, 2\}$
$\text{List}_G : \{13, 7, 5^2, 5^2, 3^3, 3, 3, 3, 2^2, 2^2, 2, 2\}$
$\text{List}_1$(after cancellation): $\{13, 3^3, 3, 2^2, 2^2, 2\}$
$\text{Inv}_1 : 13 \cdot 3^3 \cdot 2^2$
$\text{List}_2 : \{3, 2^2, 2\}$
$\text{Inv}_2 : 3 \cdot 2^2$
$\text{List}_3 : \{2\}$
$\text{Inv}_3 : 2$
$\text{List}_4 : \{\} \text{ So now we're done.}$
Then $\text{Inv}(K) = \{13 \cdot 3^3 \cdot 2^2, 3 \cdot 2^2, 2\}$