

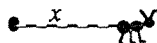
Mathematics Support Capsules

Related Rates

by Raymond W. Bacon

A **variable** is something that changes.

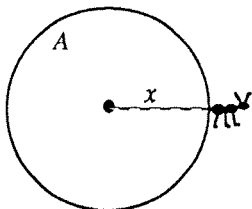
If you have a variable, then you probably have a **rate** of change. For instance, x could be how far an ant is from his hole.



Then $\frac{dx}{dt}$ would be the rate of change of x with respect to t , i.e. how fast he's going. He might, for instance, be going 2in./sec. $\left(\frac{dx}{dt} = 2\frac{\text{in.}}{\text{sec}}\right)$

If you have two variables that are **related**, then you probably can **relate** their rates.

For example, the area of the circle shown to the right is related to x by the well-known formula $A = \pi x^2$; you tell me how far out the ant is, and I'll tell you the area of the circle. If $x = 2/\sqrt{\pi}$ ft., then $A = 4$ ft.².



Now, what happens as the ant moves? x changes and that causes A to change. The **rate** of change of A is **related** to the rate of change of x . How?

Take the derivatives of both sides of the equation with respect to t .

$$A = \pi x^2 \quad \text{related variables}$$

$$\frac{dA}{dt} = 2\pi x \frac{dx}{dt} \quad \text{related rates}$$

Now, you tell me how fast the ant is moving, and I'll tell you how fast the area of the circle is changing. For instance, if $\frac{dx}{dt} = 2\frac{\text{in.}}{\text{sec.}}$ when the ant is 5 inches from the hole, then $\frac{dA}{dt} = 2\pi(5)(2) = 20\pi\frac{\text{in.}^2}{\text{sec.}}$

Note the **when** in the last sentence. There is typically a **when**, usually a final value given in the problem. **WARNING:** Don't put in the **when** until the end of the problem. Otherwise you'll take derivatives of constants and get **nothing** where there is supposed to be **something**.

Of course, in order to get the related rates correctly, you have to be able to take derivatives correctly. Let's review quickly.

First, decide what variable you're taking the derivative with respect to. Usually it's t in related rates problems, but not always.

$$\text{If } y = t^2, \text{ then } \frac{dy}{dt} = 2t \text{ is easy to see.}$$

$$\text{If } y = x^2, \text{ then } \frac{dy}{dt} = 2x \frac{dx}{dt} \text{ is sometimes not so easy to see.}$$

[RULE OF THUMB: If the thing you're taking the derivative of (x^2 in this case) has a variable in it which is not the same as the variable you're differentiating with respect to (t in this case), then you'll get a derivative (dx/dt in this case) on the end.]

Remember the formula for the derivative of $y = u^n$, where u is a function of x :

$$\frac{dy}{dx} = nu^{n-1} \frac{du}{dx}$$

The $\frac{dx}{dt}$ above is like the $\frac{du}{dx}$ in the formula. It comes from the chain rule, really.

Make sure your dx/dt is there, or you won't have a rate to relate to!

Watch out when things get switched around:

$$\text{when } t = y^3, \text{ then } \frac{dt}{dx} = 3y^2 \frac{dy}{dx}$$

Here's a harder one:

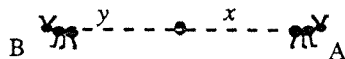
$$V = \frac{\pi r^2}{3} h$$

$$\text{So } \frac{dV}{dt} = \frac{\pi}{3} \left(r^2 \frac{dh}{dt} + h(2r) \frac{dr}{dt} \right)$$

If you're having trouble taking the derivatives in related rates problems, go back and practice some implicit differentiation problems from before.

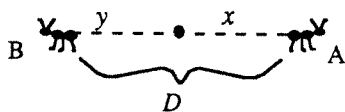
Let's go on with our ant problem. Suppose we have two ants, A and B. Ant A is walking to the **right** at the rate of 2 mm/sec. (Start drawing a picture.) So

$\frac{dx}{dt} = 2$ mm/sec. Ant B is walking to the **left** at the rate of 3 mm/sec. So you **invent** the symbol y for distance to the left and say $\frac{dy}{dt} = 3$ mm/sec.



The question is: How fast is the distance between the ants changing?

First, invent the symbol D and put it on the picture. The question rephrased in symbols is: $\frac{dD}{dt} = ?$



So D , x , and y are the variables, related by $D = x + y$. Now take the derivative of both sides with respect to t :

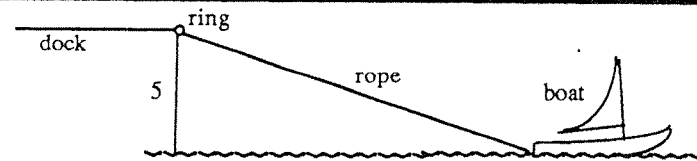
$$\frac{dD}{dt} = \frac{dx}{dt} + \frac{dy}{dt} = 2 + 3 = 5 \text{ mm/sec.}$$

That's it. (There was no **when** in this problem, but there usually is.)

Now let's do a harder one.

Note that most related rates problems, like max-min problems, are "word problems". It is very important to **READ** the problem carefully before doing anything else. Word problems are probably the hardest problems in calculus.

A boat five feet below the top of a dock is being pulled in by a rope through a ring on the top of the dock. The rope is being pulled in at a rate of 3 ft./sec. How fast is the boat approaching the dock when it is twenty feet away?



Notice that we can put in the 5 **now**, because it is **not a variable**, but **not the 20**, because the distance from the dock **does change**.

The 20 is the **when** and gets put in only at the end.

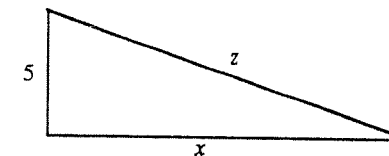
As the boat moves toward the dock, the length of rope changes, so **INVENT A VARIABLE**, say z , for rope length and write down its given rate of change.

$$\frac{dz}{dt} = -3 \frac{\text{ft.}}{\text{sec.}} \quad (\text{negative because } z \text{ is decreasing})$$

The distance from the boat to the dock changes. Call this x . Now,

↓↓↓ **RESTATE THE QUESTION** ↓↓↓ :

$$\frac{dx}{dt} = ? \text{ when } x = 20.$$



Note that we have simplified the drawing to its mathematically interesting elements. This helps to get rid of extraneous, possibly confusing, information.

Now determine which variables are related and **WRITE AN EQUATION** relating them (choose the form of the equation that is easiest to differentiate).

$$x^2 + 5^2 = z^2 \quad (\text{although } z = \sqrt{x^2 + 5^2} \text{ would have worked also})$$

$$\text{DIFFERENTIATE: } 2x \frac{dx}{dt} + 0 = 2z \frac{dz}{dt} \quad (\text{derivative of a constant is zero})$$

↓↓↓ **SOLVE FOR THE UNKNOWN RATE** ↓↓↓

$\frac{dx}{dt} = \frac{z dz}{x dt} = \frac{\sqrt{425}}{20} (-3) \text{ ft./sec.}$ (throw in the units at the end)

(given)

(a hidden part of the **when**. You need z when $x = 20$. In this case, go back to the equation and put in $x = 20$ to get $20^2 + 5^2 = z$. Solve.)

(here's where you plug in the **when**.)

Look at your answer and make sure it makes sense. dx/dt is negative because x is decreasing (because the boat is getting closer to the dock). Check the magnitude of the answer:

$$\frac{\sqrt{425}}{20}(3) = \frac{\text{something} > 20}{20}(3) = (\text{something} > 1)(3) = \text{a little more than } 3.$$

This seems reasonable. If you pull in the rope at 3 ft./sec., you don't expect the boat to come in at the speed of light.

We now have a systematic method of attack for related rates problems:

0. **READ.** Read the problem **several times**. Read it until you **understand** it intuitively. The problem has to make sense to you before you can conceive of a solution.
1. **DRAW.** **Always** make a sketch. The problem will make sense to you if you can draw a picture of it. Most of the time your **idea for a solution** will come from your **picture**.
2. **INVENT SYMBOLS** and **RESTATE THE QUESTION** in terms of the symbols on the drawing.
3. **WRITE DOWN AN EQUATION** in the variables you are relating.
4. **DIFFERENTIATE** with respect to the correct variable.
5. **SOLVE FOR THE UNKNOWN RATE** and plug in the **when**.
6. **CHECK.** Does the answer seem reasonable? Does it make sense? Does it answer the question?

A conical container of depth 2 ft. and radius 8 in. is draining in such a way that its fluid depth is changing at the rate of 6 in./min. into a cylindrical container below it of height 2 ft. and radius 8 in. If the cone was full at the beginning, then how fast is the depth in the cylinder changing when the depth in the cone is 1 ft.?

↓↓↓ INVENT SYMBOLS ↓↓↓

V_t = vol. of fluid in top

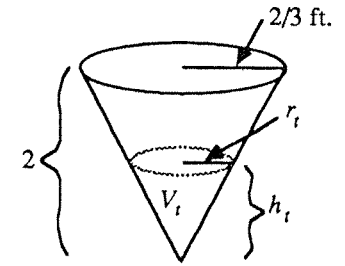
V_b = vol. of fluid in bottom

r_t = radius in top

r_b = radius in bottom

h_t = height in top

h_b = depth in bottom

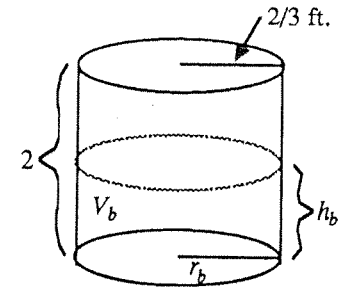


$$\frac{dh_t}{dt} = -\frac{1}{2} \text{ ft./min.}$$

(the given rate)

↓↓↓ RESTATE THE QUESTION ↓↓↓

$$\frac{dh_t}{dt} = ? \text{ when } h_t = 1$$



We need an equation relating h_b and h_t . These two variables have to do with V_b and V_t . So, relate the volume of liquid in the top to the volume of fluid in the bottom, because we have formulas for the volumes of cones and cylinders: ($V_{\text{cone}} = \frac{\pi}{3}r^2h$; $V_{\text{cyl}} = \pi r^2h$).

Volume in bottom = what has already flowed out of top

V_b = total volume of cone at beginning minus what's left in top

$$V_b = V_{\text{cone}} - V_t$$

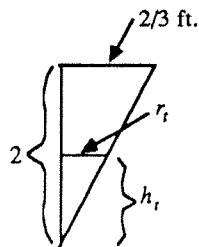
$$\pi(r_b)^2h_b = \frac{\pi}{3}(2/3)^2(2) = \frac{\pi}{3}(r_t)^2(h_t)$$

$$r_b = 2/3$$

$$\pi(2/3)^2h_b = \frac{\pi}{3}(2/3)^2(2) = \frac{\pi}{3}(r_t)2(h_t)$$

Now, if I can get rid of r_t , I'll be ready to differentiate. I want to find r_t in terms of either h_t or h_b or both. By similar triangles in the cone: $\frac{h_t}{r_t} = \frac{2}{2/3} = 3$, so $r_t = \frac{h_t}{3}$. [It is common to find a second equation so you can substitute into the first to get an equation in only the variables you're interested in.]

$$\text{Now, } \pi(2/3)^2 h_b = \frac{\pi}{3} (2/3)^2 (2) - \frac{\pi}{3} \left(\frac{h_t^3}{9}\right)$$



TAKE DERIVATIVES (with respect to t)

$$\pi(2/3)^2 \frac{dh_b}{dt} = -\frac{\pi}{27} (3h_t^2) \frac{dh_t}{dt}$$

$$\text{So } \frac{dh_b}{dt} = \frac{-\pi 3(h_t)^2 (dh_t/dt)}{27\pi(2/3)^2} . \text{ Now plug in values } h_t = 1 \text{ and } (dh_t/dt) = -(1/2).$$

$$\text{Then } \frac{dh_b}{dt} = -\frac{(1)(-1/2)}{4} = +\frac{1}{8} \text{ ft./min.}$$

CHECK: Does $+\frac{1}{8}$ ft./min. seem reasonable? The plus sign is right. The depth in the top is decreasing and the depth in the bottom is increasing (i.e. "plus"). The cylinder is bigger so the rate of increase in depth will be less than the rate of decrease in the cone.

FINAL WORD: Related rates problems are **hard**. They are missed on tests (along with max-min problems) more often than any other type of problem. There is only one way to learn how to do them: **DO THEM YOURSELF**. When somebody shows you how to do one, it always looks easy. That's because you didn't have to do the set-up yourself: the **READING**, the **DRAWING**, **INVENTING THE VARIABLES**, figuring out what the question is, deciding what principle to use to set up the equations, etc. The only way you can expect to set one up yourself under pressure on a test is if you have worked out 15 or 20 of them yourself already beforehand. So, turn to the book, start with easy ones and do lots of them. Learn the tricks and traps for yourself. Check with your instructor, TA, or tutor right away if you get stuck. Gain experience. It will pay off.

EXERCISES

to accompany

Mathematics Learning Module IV: Related Rates

compiled by

Mathematics Support Center, 8/81

- A. Sand falls onto a conical pile at the rate of 10 ft.³/min. The radius of the base of the pile is always equal to one-half of its altitude. How fast is the altitude of the pile increasing when it is 5 ft. deep?

ANS: $8/5\pi$ ft./min.

- B. A spherical balloon is inflated with gas at the rate of 100 ft.³/min. Assuming that the gas pressure remains constant, how fast is the radius of the balloon increasing at the instant when the radius is 3 ft.?

ANS: $25/9\pi$ ft./min.

- C. A boat is pulled in to a dock by means of a rope with one end attached to the bow of the boat, the other end passing through a ring attached to the dock at a point 4 ft. higher than the bow of the boat. If the rope is pulled in at the rate of 2 ft./sec., how fast is the boat approaching the dock when 10 ft. of rope is out?

ANS: $10/\sqrt{21}$ ft./sec.

- D. Assuming that a soap bubble retains its spherical shape as it expands, how fast is its radius increasing when its radius is 2 inches, if air is blown into it at the rate of 4 in.³/sec.?

ANS: $1/4\pi$ in./sec.

- E. A student is using a straw to drink from a conical paper cup, whose axis is vertical, at the rate of 6 in.³/sec. If the height of the cup is 10 inches and the diameter of its opening is 6 inches, how fast is the level of the liquid falling when the cup is half full?

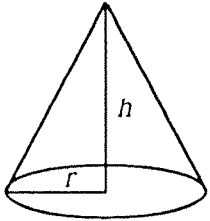
ANS: Half Vol. $\rightarrow h = \sqrt[3]{500}$, so $\frac{dh}{dt} = \frac{-4}{3\pi\sqrt[3]{2}}$ in./sec.

- F. (not easy!) Two 60 ft. street lights are 100 ft. apart. The light at the top of one is functioning but the other is being repaired by a worker. If the worker drops a tool kit from the top of the second pole, how fast is its shadow moving when the kit is 20 ft. from the ground? (Use $s=16t^2$ for the distance the kit falls in t seconds.)

ANS: $60/\sqrt{10}$ ft./sec.

ANSWERS
to EXERCISES
Mathematics Learning Module IV: Related Rates

A)



Given: $\frac{dV}{dt} = 10 \text{ ft.}^3/\text{min.}$

$h = 2r$ or $r = \frac{1}{2}h$

$V = \frac{1}{3}\pi r^2 h$

Solve: $\left. \frac{dh}{dt} \right|_{h=5}$

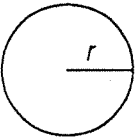
(this means $\frac{dh}{dt}$ when $h = 5$)

So: $V = \frac{1}{3}\pi r^2 h$
 $= \frac{1}{3}\pi \left(\frac{1}{2}h\right)^2 h$
 $= \frac{1}{12}\pi h^3$

Then: $\frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}$
 $10 = \frac{1}{4}\pi h^2 \frac{dh}{dt}$
 $\frac{dh}{dt} = \frac{40}{25\pi}$
 $= \frac{8}{5\pi} \text{ ft./min.}$

Comments: In problems involving cones, you will be given a relationship between r and h (e.g. $2r = h$). Look for them. Always substitute non-constants only after differentiating.

B)



Given: $\frac{dV}{dt} = 100 \text{ ft.}^3/\text{min.}$

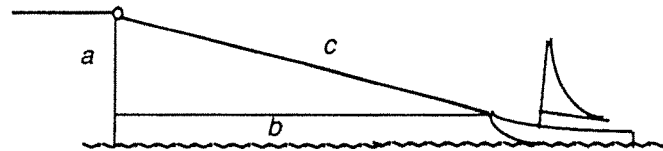
$V = \frac{4}{3}\pi r^3$

Find: $\left. \frac{dr}{dt} \right|_{r=3}$

So: $V = \frac{4}{3}\pi r^3$
 $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$
 $100 = 4\pi(3)^2 \frac{dr}{dt}$
 $\frac{dr}{dt} = \frac{100}{36\pi}$
 $= \frac{25}{9\pi} \text{ ft./min.}$

Comments: When a change is given as $\text{ft.}^3/\text{min.}$, it is always $\frac{dV}{dt}$.

C)



Given: $a = 4$ (a constant)

$\frac{dc}{dt} = 2 \text{ ft./sec.}$

$a^2 + b^2 = c^2$

Find: $\left. \frac{db}{dt} \right|_{c=10}$

So: $a^2 + b^2 = c^2$
 $4^2 + b^2 = c^2$

$0 + 2b \frac{db}{dt} = 2c \frac{dc}{dt}$

When $c = 10$, $a = 4$ (constant)

$b = \sqrt{c^2 - a^2}$

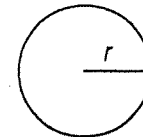
$= \sqrt{84} = 2\sqrt{21}$

Then: $2(2\sqrt{21}) = 2(10)(2) \frac{db}{dt}$

$\frac{db}{dt} = \frac{10}{\sqrt{21}} \text{ ft./sec.}$

Comments: $a = 4$ is a constant because the height from the bow of the boat to the dock will not change. Only the distance between the boat and the dock will change.

D)



Given: $\frac{dV}{dt} = 4 \text{ in.}^3/\text{sec.}$

$V = \frac{4}{3}\pi r^3$

Find: $\left. \frac{dr}{dt} \right|_{r=2}$

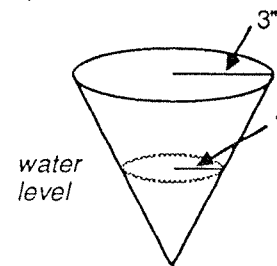
So: $V = \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

$4 = 4\pi(2)^2 \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{4}{4\pi 4} = \frac{1}{4\pi} \text{ in./sec.}$

E)



Use: t = time in seconds
 $r(t)$ = radius of top of water-level circle at time t (in inches)
 $h(t)$ = depth of water (in inches)
 $V(t)$ = volume of water (in. 3) at time t

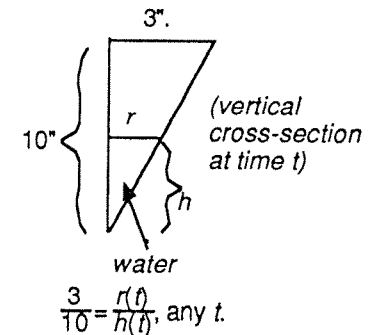
Given: $\frac{dV}{dt} = -6$

Wanted: $\frac{dh}{dt}$ at moment when cup is half full.

Full Volume of Cup: $V = \frac{1}{3}\pi(3)^2(10)$
 $= 30\pi \text{ in.}^3$

Cup Half Full: $V = 15\pi \text{ in.}^3$

Since cone of water is of same shape as cone holding the water, the two triangles shown in the vertical cross-section must be similar:



Then: $r(t) = \frac{3}{10}h(t)$.

Thus, at any time t , $V(t) = \frac{1}{3}\pi(\frac{3}{10}h(t))^2h(t) = \frac{3\pi}{100}(h(t))^3$

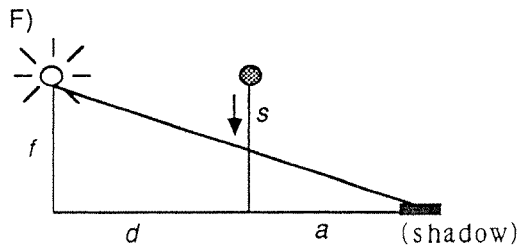
So: When the cup is half full, h is found by

$$15\pi = \frac{3\pi}{100}h^3 \Rightarrow h^3 = 500 \Rightarrow h = \sqrt[3]{500} \text{ inches}$$

Wanted: $\frac{dh}{dt}$ when $h = \sqrt[3]{500}$ inches

Chain Rule: $-6 = \frac{dV}{dt} = \frac{3\pi}{100}3h^2\frac{dh}{dt}$ (when $h = \sqrt[3]{500}$ -- plug in data)

$$= \frac{9\pi}{100}(\sqrt[3]{500})^2\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \boxed{\frac{-4}{3\pi\sqrt{2}} \text{ in./sec.}}$$



Given: $f = 60$; $d = 100$ (constants)
 $c = 60 - s$
 $s = 16t^2$

Find $\frac{dc}{dt}$: $c = 60 - s$, $s = 16t^2$
 $\therefore \frac{dc}{dt} = -32t$

Find: $\frac{da}{dt}|_{c=20}$

Find a when $c = 20$:
 $c(a + 100) = ca + 100c$
 $20a + 100(20) = 60a$
 $40a = 100(20) \Rightarrow a = 50$

Now, $\frac{c}{a} = \frac{f}{d+a}$ (sim. triangles)
 $= \frac{60}{100+a}$

Find t when $c = 20$:
 $16t^2 = s = 60 - c = 60 - 20 = 40$
 $\therefore t = \sqrt{\frac{40}{16}} = \frac{\sqrt{10}}{2}$

So $100c + ca = 60a$

$$100\frac{dc}{dt} + c\frac{da}{dt} + a\frac{dc}{dt} = 60\frac{da}{dt}$$

Substitute the values found for t , $\frac{dc}{dt}$, and a in the equation labelled (**):
 $\frac{da}{dt} =$

$$\frac{dc}{dt}(100 + a) = \frac{da}{dt}(60 - c)$$

$$\frac{da}{dt} = \frac{dc}{dt}(100 + a)\left(\frac{1}{60 - c}\right)$$

(call this equation **)

$$-32\left(\frac{\sqrt{10}}{2}\right)(100 + 50)\left(\frac{1}{60 - 20}\right)$$

$$= \boxed{-60\sqrt{10} \text{ ft./sec.}}$$

Comments: Answer is negative because the object that makes the shadow is falling.