MULTIPLE INTEGRALS

I. Double Integrals:

(1) Evaluate each of the following double integrals, and sketch the region \( A \) over which the integration extends.

(a) \[ \int_0^1 \int_0^x \sin y \, dy \, dx \]
(b) \[ \int_0^1 \int_0^y \sin x \, dx \, dy \]
(c) \[ \int_0^1 \int_0^1 y^2 \, dx \, dy \]
(d) \[ \int_0^1 \int_0^1 ye^x \, dx \, dy \]
(e) \[ \int_0^1 \int_0^1 \frac{1}{x^2 + y^2} \, dx \, dy \]
(f) \[ \int_0^1 \int_0^1 \frac{1}{x^2 + y^2} \, dx \, dy \]

(2) Evaluate \( \int \int_R \, dA \) where \( R \) is the region between \( y = 2x \) and \( y = x^2 \) lying to the left of \( x = 1 \).

(3) Find the area of the region bounded by the parabola \( x = y - y^2 \) and the line \( xy = 0 \).

(4) Using polar coordinates and double integration find (a) the total area enclosed by the lemniscate \( r^2 = 2a^2 \cos 2\theta \) and (b) the area that lies inside the cardioid \( r = a(1+\cos \theta) \) and outside the circle \( r = a \).

(5) Find the center of gravity (letting \( \delta(x,y) = 1 \)) of the area bounded by the coordinate axes and the line \( xy = a \).

(6) Find the center of gravity of the area bounded by the curve \( y^2 + x = 0 \) and the line \( y = x^2 \) (letting \( \delta(x,y) = 1 \)).

(7) Find the moment of inertia \( (\delta = 1) \) about the \( x \)-axis of the area bounded by the curve \( y = e^x \) and the lines \( x = 0, x = 1 \) (letting \( \delta = 1 \)).

(8) Find the moment of inertia about the \( z \)-axis of the area bounded by the \( x \)-axis, the curve \( y = e^x \) and the lines \( x = 0, x = 1 \) (letting \( \delta = 1 \)).

(9) Using polar coordinates, find the moment of inertia \( I_0 \) with respect to an axis through \( O \) perpendicular to the \( xy \)-plane for the area lying inside the cardioid \( r = a(1+\cos \theta) \) and outside the circle \( r = a \).

(10) Using double integration, find the following volumes: (a) in the 1st octant between the planes \( z = 0 \) and \( z = x+y+2 \) and inside the cylinder \( x^2 + y^2 = 16 \). (b) bounded by the cylinder \( x^2 + y^2 = 4 \) and the planes \( y+z = 4 \) and \( z = 0 \). (c) bounded above by the paraboloid \( x^2 + y^2 = z \), below by the plane \( z = 0 \), and laterally by the cylinders \( y^2 = x \) and \( x^2 = y \). (d) the wedge cut from the cylinder \( 4x^2 + y^2 = a^2 \) by the planes \( z = 0 \) and \( z = my \).

II. Triple Integrals:

(1) By triple integration find the following volumes:

(a) of the tetrahedron bounded by the plane \( x/a + y/b + z/c = 1 \) and the coordinate planes.

(b) between the cylinder \( z = y^2 \) and the \( xy \)-plane that is bounded by the four vertical planes \( x = 0, x = 1, y = -1, y = 1 \).

(c) in the 1st octant bounded by the cylinder \( x = y^2 \) and the planes \( z = y, x = 0, z = 0 \).

(d) Enclosed by the cylinder \( y^2 + 4z^2 = 16 \) and the planes \( x = 0, x+y = 4 \).

(e) Inside \( x^2 + y^2 = 9 \), above \( z = 0 \) and below \( x+z = 4 \).

(2) Using cylindrical coordinates, find the volume:

(a) bounded above by the paraboloid \( z = 5-x^2-y^2 \) and below by the paraboloid \( z = 4x^2+4y^2 \).

(b) that is bounded above by the paraboloid \( z = 9-x^2-y^2 \) below by the \( xy \)-plane, and that lies outside the cylinder \( x^2+y^2 = 1 \).

(c) bounded below by the paraboloid \( z = x^2+y^2 \) and above by the plane \( z = 2 \).

(d) bounded above by the sphere \( x^2+y^2+z^2 = 2a \) and below by the paraboloid \( ax = z^2+2z \).

(3) Using spherical coordinates find the volume

(a) of the solid which lies above the cone \( z = x^2+y^2 \) and inside the sphere \( x^2+y^2+z^2 = 4a \).

(b) cut from the sphere \( \rho = 2 \) by the plane \( z = \sqrt{2} \).

(c) enclosed by the surface \( \rho = a \) (letting \( \phi \)).

III. Applications of Triple Integration:

(1) Find the volume and centroid of the solid bounded by the graphs of \( z = x^2+y^2, x^2+y^2 = 4, \) and \( z = 0 \).

(2) Find the moment of inertia of a homogeneous circular cylinder of altitude \( h \) and radius of base \( a \) with respect to each of the following:

(a) the axis of the cylinder.

(b) the diameter of the base.
(3) Find the mass and center of mass of a solid hemisphere of radius \( r \) if the density at a point \( P \) is directly proportional to the distance from the center of the base to \( P \).

(4) Find the moment of inertia with respect to the axis of the hemisphere in the above problem.

(5) Find the moment of inertia about the \( x \)-axis for the volume cut from the sphere \( x^2 + y^2 + z^2 = 4a^2 \) by the cylinder \( x^2 + y^2 = a^2 \).

(6) Use cylindrical coordinates to find the moment of inertia of a sphere of radius \( a \) and mass \( M \) about its diameter.

(7) Find the moment of inertia of a rt. circular cone of base radius \( a \), altitude \( h \), and mass \( M \) about an axis through the vertex and parallel to the base.

(8) Find the center of gravity of the volume (which resembles a filled ice-cream cone) that is bounded above by the sphere \( r = a \) and below by the cone \( \theta = \pi/6 \).

(9) Find the radius of gyration with respect to a diameter of a spherical shell of mass \( M \) bounded by the spheres \( r = a \) and \( r = 2a \) if the density is \( \delta = \rho \).

I. Double Integrals:

1. a) \( \int_0^a \int_0^{\pi/2} r \sin y \, r \, dr \, dy = \int_0^a \left[ \frac{-r \cos y}{2} \right]_0^a \, dy = \frac{\pi a^2}{2} \)

1. b) \( \int_0^a \int_0^a r \sin \theta \, r \, dr \, d \theta = \int_0^a r \sin \frac{\pi}{2} \, dr = \frac{r^2}{2} \left[ \frac{x}{\sin y} \right]_0^\pi = \frac{\pi a^2}{2} \)

1. c) \( \int_0^a \int_0^a x \, dx \, dy = \int_0^a \left[ \frac{1}{2} (x^2 + xy \sin y) \right]_0^a \, dy = \frac{a^3}{2} \)

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\begin{align*}
\text{(a)} & \quad \int_0^a \int_0^{\pi/2} r \sin y \, r \, dr \, dy = \int_0^a \left[ \frac{-r \cos y}{2} \right]_0^a \, dy = \frac{\pi a^2}{2} \\
\text{(b)} & \quad \int_0^a \int_0^a r \sin \theta \, r \, dr \, d \theta = \int_0^a r \sin \frac{\pi}{2} \, dr = \frac{\pi a^2}{2} \\
\text{(c)} & \quad \int_0^a \int_0^a x \, dx \, dy = \int_0^a \left[ \frac{1}{2} (x^2 + xy \sin y) \right]_0^a \, dy = \frac{a^3}{2}
\end{align*}
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\[M = \iiint x \, dV = \frac{2}{3} \pi a^3 \]

\[m = \iiint \rho \, dV = \frac{2}{3} \pi a^3 \]

\[\bar{x} = \frac{M}{m} \rho = \frac{2}{3} \pi a^3 \]

\[\bar{y} = \frac{M}{m} \rho = \frac{2}{3} \pi a^3 \]

\[\bar{z} = \frac{M}{m} \rho = \frac{2}{3} \pi a^3 \]

\[
\begin{align*}
\bar{x} & = \frac{Mx}{m} = \frac{2}{3} \pi a^3 \rho = \frac{2}{3} a^3 \\
\bar{y} & = \frac{Mx}{m} = \frac{2}{3} \pi a^3 \rho = \frac{2}{3} a^3 \\
\bar{z} & = \frac{Mx}{m} = \frac{2}{3} \pi a^3 \rho = \frac{2}{3} a^3
\end{align*}
\]

\[\vec{r} = (\bar{x}, \bar{y}, \bar{z}) = (\frac{2}{3} a^3, \frac{2}{3} a^3, \frac{2}{3} a^3)\]