LOGARITHMS

I. BASE 10

II. BASE b

III. PROPERTIES OF LOGARITHMS

IV. MORE PROPERTIES AND CHANGING BASES

V. SOLVING LOGARITHMIC AND EXPONENTIAL EQUATIONS

VI. GRAPHING

VII. SUMMARY, FOR NATURAL LOGARITHMS

I. LOGARITHMS BASE 10

\[ \log_{10} x \] is the number to which you must raise 10 to get \( x \).

so: \[ \log_{10} 10^y = y \]

\( y \) is called the logarithm of \( x \), base 10. Notice that \( y \) is an exponent.

Think of it this way:

\[ \log_{10} x = y \] following the arrows we get \( 10^y = x \)

examples:

1. \[ \log_{10} 100 = 2 \] since \( 10^2 = 100 \)

2. \[ \log_{10} .001 = -3 \] since \( 10^{-3} = .001 \)

in example 1, the logarithm is \( 2 \).

in example 2, it's \( -3 \).

Exercises:

1) \[ \log_{10} 1000 = ? \]

2) \[ \log_{10} .01 = ? \]

3) \[ \log_{10} 10 = ? \]

4) \[ \log_{10} 1 = ? \]

5) \[ \log_{10} 10^{25} = ? \]

SOLVE FOR \( x \):

6) \[ \log_{10} x = 5 \] \( x = ? \)

7) \[ \log_{10} x = 1/2 \] \( x = ? \)

8) \[ \log_{10} x = 0 \] \( x = ? \)

9) \[ \log_{10} x = -1 \] \( x = ? \)

10) \[ \log_{10} 3x = 1 \] \( x = ? \)

Answers:

1) 3

2) -2

3) 1

4) 0

5) 25

6) \( 10^5 \) or 100,000

7) \( 10^{1/2} \) or \( \sqrt{10} \)

8) 16 or 1

9) 10 or 1/10

10) 2
II. LOGARITHMS BASE b

\[ \log_b x \] is the number to which you must raise b to get x. b can be any positive real number except 1.

so: \( \log_b x = y \) means \( b^y = x \)

Y is called the logarithm of X, base b. Notice that Y is an exponent.

Think of it this way:

\[ \log_b x = y \] follow the arrows to get:

\[ b^y = x \]

examples:
1. \( \log_2 8 = 3 \) since \( 2^3 = 8 \)
2. \( \log_3 (1/9) = -2 \) since \( 3^{-2} = 1/9 \)
3. \( \log_4 2 = 1/2 \) since \( 4^{1/2} = \sqrt{4} = 2 \)

in example 1, the logarithm is 3.

in example 2, it's -2.

in example 3, it's 1/2.

Exercises:

SOLVE FOR X:
1) \( \log_5 25 = ? \)
2) \( \log_9 3 = ? \)
3) \( \log_2 (1/16) = ? \)
4) \( \log_{1/2} (1/4) = ? \)
5) \( \log_{1/2} 4 = ? \)

6) \( \log_7 x = -2 \)
7) \( \log_{1/2} x = 3 \)
8) \( \log_{1/4} x = -2 \)

IIII. PROPERTIES OF LOGARITHMS

NOTE: If you see \( \log x \) with no base b, it means the base is 10.

So: \( \log x = \log_{10} x \)

Recall the properties of exponents:
1. \( x^m \cdot x^n = x^{m+n} \)
2. \( x^m / x^n = x^{m-n} \)
3. \( (x^m )^n = x^{mn} \)
4. \( x^0 = 1 \)
5. \( x^1 = x \)

For explanation of these properties, see capsule on EXPONENTS.

Since the \( \log \) of a number is the exponent of the number, we get the following properties of logarithms:

1. \( \log_b (XY) = \log_b X + \log_b Y \)
2. \( \log_b (X/Y) = \log_b X - \log_b Y \)
3. \( \log_b x^n = n \log_b x \)
4. \( \log_b 1 = 0 \) (since \( b^0 = 1 \))
5. \( \log_b b = 1 \) (since \( b^1 = b \))

NOTE: YOU CANNOT TAKE THE LOG OF A NEGATIVE NUMBER OR OF ZERO:

example: \( \log_{10} (-5) \) IS NOTDEFINED.

EXAMPLES

1) Simplify as much as possible: \( \log_{b} \left( \frac{x^2 y}{z^3} \right) \)

\[ \log_{b} \left( \frac{x^2 y}{z^3} \right) = \log_{b} (x^2) - \log_{b} (z^3) \] using property 2

\[ = \log_{b} (x^2) + \log_{b} y - \log_{b} (z^3) \] using property 1

\[ = 2 \log_{b} x + \log_{b} y - 3 \log_{b} z \] using property 3

Answers:
1) 2
2) 1/2
3) -4
4) 2
5) -2
6) \( y^2 \) or 1/49
7) \( (\frac{y}{z})^2 \) or 1/8
8) \( (\frac{y}{z})^{-2} \) or 1/49
Examples cont’d.

2) Put into one logarithm: 
\[ \log_5 9 + 3 \log_5 2 - 3 \log_5 3 \]

\[ \Rightarrow \log_5 (9 \cdot 2^3) - 3 \log_5 3 \]

\[ \Rightarrow \log_5 (\frac{9 \cdot 2^3}{3^3}) \]

\[ \Rightarrow \log_5 (\frac{12}{8}) = \log_5 (\frac{3}{2}) \]

[Note that the base must be the same in each term.]

Using property 3

Using property 1

Using property 2

Multiplying together

EXERCISES

I. Simplify

a) \( \log_2 (2x) \)
b) \( \log_b (\frac{xy}{z})^{1/2} \)
c) \( \log_{10} (\sqrt[3]{x^y}) \)
d) \( \log (x^2 - 1) \)

II. Put into a single Logarithm

a) \( \frac{1}{2} (\log_b x + \log_b y) \)
b) \( 4 \log 3 - 2 \log 3 \)
c) \( 2 \log 3 - \frac{1}{2} \log 9 \)
d) \( 10(\log_b X - \log_b Y - \log_b z) \)

Answers

I. a) \( \log_2 2 \)
b) \( \frac{3}{2} \log_2 x + \frac{3}{2} \log_2 y - \frac{3}{2} \log_2 z \)
c) \( \frac{2}{3} \log_5 x + \frac{2}{3} \log_5 y \)
d) \( \log (x+1) = \log (x+1) \)

II. a) \( \log_b (xy)^{1/2} \)
b) \( \log 9 \)
c) \( \log 3 \)
d) \( \log_b (x/y)^{1/2} \)

IV. More Properties of Logarithms and Change of Base

A. If \( X = Y \), then \( \log_b X = \log_b Y \)

B. If \( \log_b X = \log_b Y \), then \( X = Y \)

C. \( \log_a x = \frac{\log_b x}{\log_b a} \) (See an algebra or calculus text for explanation of this formula to change from \( \log_b x \) to \( \log_a x \).

D. \( \log_b x = x \) (since \( \log_b x \) is what you must raise \( b \) to, to get \( x \).

E. \( \log_b b^m = m \) (since \( \log_b b^m = m \log_b b = m \cdot 1 = m \)

We’ll use these properties in the next section on solving equations.
V. SOLVING LOGARITHMIC EQUATIONS

Examples:

1) Solve for \( x \): \( 5^x = 6 \)

   we can solve two different ways:

   1) NOTICE \( 5^x = 6 \) means \( \log_5 6 = x \) using logarithmic notation.

   2) OR you can just take the log of both sides, using property A: Since \( 5^x = 6 \), we know \( \log_b 5^x = \log_b 6 \) (we can use any "b" we want)

   NOW:

   \[ \log_b 5^x = \log_b 6, \text{ and} \]

   \[ x \log_b 5 = \log_b 6 \text{ using property 3} \]

   \[ \text{so: } x = \frac{\log_b 6}{\log_b 5} \]

   \( \text{Note: If we plot } b^x \text{, then } x = \frac{\log_b b^x}{\log_b b} = \frac{x}{1} \text{ (since } \log_b b = 1 \text{)} \)

   \( \text{If we plot } b^x \text{, then } x = \frac{\log_b b^x}{\log_b b} = \frac{x}{1} \text{ (since } \log_b b = 1 \text{)} \)

2) Solve for \( x \): \( 2^{x-1} = 5^x \)

   1: take logs of both sides using \( b=2 \) so:

   \[ \log_2 2^{x-1} = \log_2 5^x \text{ (using A)} \]

   2: \( (x-1) \log_2 2 = x \log_2 5 \) using property 3

   3: \( (x-1) = x \log_2 5 \) using property 5

   4: \( x - x \log_2 5 = 1 \) get all the \( x \)-terms on one side.

   5: \( x(1-\log_2 5) = 1 \) factor out the \( x \).

   6: \( x = \frac{1}{1 - \log_2 5} \) divide

   \( \text{NOTE: } \text{YOU CAN ALSO USE } b = 5 \text{ here. Try it!} \)

Examples cont'd.

3. Simplify: \( 10^2 \log 5x \) (No b here since base = 10).

   We know that \( 10 \log_{10} a = a \) by property 0; so we'd like to get \( 10^2 \log 5x \) in this form.

   \( \text{Well: } 2 \log 5x = \log(5x)^2 \) by 3

   \( \text{So: } 10^2 \log 5x \)

   \( \log(5x)^2 = 10 \log 25x^2 \)

   \( = 10 \log 25x^2 = 25 \log 25 \)

   \( = 25 \) by D.

4. Solve for \( x \): \( \log_b 4x + \log_b 2 = 2 \)

   \( \log_b (4x + 2) = 2 \) using property 1

   convert to exp. form \( \log_b (8x) = 2 \)

   \( \text{so } 8^2 = 8x \)

   \( \text{so } x = 8 \)

   Strategy: Aim for a single logarithm on the left so you can convert to exponential form.
EXERCISES

1. Solve for x:
   a) $7^x = 15$
   b) $5^{x-1} + 5 = 30$
   c) $3^x = 4^{x-1}$
   d) $3^2x = 9^{3x-4}$
   e) $\log_2 x - \log_2(x+4) = -5$
   f) $\log(\log x) = 3$
   g) $\log_4 (x+3) - \log_4(x-3) = 2$
   h) $\log_5 x^2 + 9 = 1$

II. Simplify:
   a) $3 \log_3 2x$
   b) $3 \log_2 x - 8 \log_2 y$
   c) $\log_b x + \log_b \left(\frac{1}{x}\right)$
   d) $10^2 + \log x$

Answers:
1. a) $x = \log_5 15$ or $\log_{10} \frac{10}{9}$ (for any b)
   b) $x = 3$
   c) $x = \frac{4 \log_2 4 - \log_2 3}{3 \log_2 9 - 2 \log_2 3}$ (for any b)
   d) $x = \frac{\log_9 9 - 2 \log_3 3}{3 \log_9 9 - 2 \log_3 3}$ (if you simplify)
   e) $x = -\frac{1}{2}$
   f) $x = 1000$
   g) $x = 5$
   h) $x + 4$ or $x = -4$

II. a) $32x^3$
   b) $x^2y^8$
   c) $1$
   d) $100x$

VI. GRAPHING EXPONENTIAL AND LOGARITHMIC FUNCTIONS

1) $y = 2^x$
   Plot points, picking points which are easy to graph. Then connect the points.
   5 points are usually good enough.

<table>
<thead>
<tr>
<th>x</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>1/4</td>
<td>1/2</td>
<td>1</td>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>

   Notice that these 2 graphs are symmetric or "mirror images" of each other across the y-axis. Also notice that y is always positive.

2) $y = (1/2)^x$

<table>
<thead>
<tr>
<th>x</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td>1/2</td>
<td>1/4</td>
</tr>
</tbody>
</table>

3) $y = \log_2 x$
   Pick values for x that are easy to compute the log of

<table>
<thead>
<tr>
<th>x</th>
<th>1/2</th>
<th>1/4</th>
<th>1</th>
<th>2</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>-1</td>
<td>-2</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

   Notice that these 2 graphs are symmetric with one another across the x-axis and y is always positive. You can't take the log of a positive number.

4) $y = \log_{1/2} x$

<table>
<thead>
<tr>
<th>x</th>
<th>1/2</th>
<th>1/4</th>
<th>1</th>
<th>2</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>-1</td>
<td>-2</td>
</tr>
</tbody>
</table>
Also: Notice that 1 and 3 and 2 and 4 are symmetric to each other across the line \( Y = X \).

\[ Y = X \]

1 and 3 are called inverses of one another.

2 and 4 are called inverses of one another.

So \( Y = 2^X \) and \( Y = \log_2 X \) are inverses of each other, also

\[ Y = (1/2)^X \]

and \( Y = \log_{1/2} X \) are inverses of each other.

The logarithm function is just the inverse of the exponential function.

For more about inverse functions, see William K. Smith, Inverse Functions (Macmillan) — it's a small paperback, and a copy has been placed in the MSC Browsing Library.

EXERCISE:

Graph: 1) \( \log Y = 10^X \) and 2) \( Y = \log_{10} X \) --- by plotting at least 4 points.

VII. SUMMARY, FOR NATURAL LOGARITHMS

NOTE: If you haven't seen logarithms base \( e \), you will see them in calculus.

4) SOLVE FOR \( x \):

\[
\begin{align*}
\ln x + \ln(x-4) &= \ln 5 \\
\ln[\ln(x-4)] &= \ln 5 & \text{by 1} \\
\text{so } e^{\ln[\ln(x-4)]} &= e^{\ln 5} \\
\text{and } [\ln(x-4)] &= 5 & \text{by 5} \\
&= x - 4 = 5 & \text{by 5} \\
x^2 - 4x - 5 &= 0 \\
(x-5)(x+1) &= 0 \\
\text{So } x &= 5, \ x = -1
\end{align*}
\]

EXAMPLES:

1) SIMPLIFY:

\[
\begin{align*}
\ln x^4 - 5 \ln y &= \frac{e^{\ln x^4}}{e^{\ln y^5}} & \text{by 7} \\
&= \frac{x^4}{y^5} & \text{by 3} \\
&= x^4 & \text{by 5}
\end{align*}
\]

2) SIMPLIFY:

\[
\begin{align*}
\ln(e^{3x/\sqrt{x}}) &= \ln e^{3x} + \ln\sqrt{x} & \text{by 1} \\
&= 3x + \ln x^{1/2} & \text{by 5} \\
&= 3x + \frac{1}{2} \ln x & \text{by 3}
\end{align*}
\]

3) \( \ln(x+y) \) does not reduce, it can't be simplified.