Introduction

Patterns Unit Overview

Intent
This unit is an essential introduction for students to the variety of ways for working on and thinking about mathematical problems. Students are introduced to general learning skills and methods that are developed and used throughout the four-year IMP curriculum and that form the foundation of the learning process through which students will build mathematical ideas. Here is a summary of these learning skills and methods.

- Working in groups to analyze problems
- Learning about group cooperation and roles in group learning
- Expressing mathematical ideas orally and in writing
- Making presentations in small groups and to the class
- Developing strategies for solving problems
- Using concrete mathematical models
- Doing investigations in which the task is not clearly defined
- Becoming familiar with alternative forms of assessment, such as self-assessment and portfolios
- Learning about the use of graphing calculators and, if available, appropriate computer software

Mathematics

*Patterns* emphasizes extended, open-ended exploration and the search for patterns. Important mathematics introduced or reviewed in *Patterns* include In-Out tables, functions, variables, positive and negative numbers, and basic geometry concepts related to polygons. Proof, another major theme, is developed as part of the larger theme of reasoning and explaining. Students’ ability to create and understand proofs will develop over their four years in IMP; their work in this unit is an important start. This unit focuses on several mathematical ideas:

- Finding, analyzing, and generalizing geometric and numeric patterns
- Analyzing and creating In-Out tables
- Using variables in a variety of ways, including to express generalizations
- Developing and using general principles for working with variables, including the distributive property
- Working with order-of-operations rules for arithmetic
Using a concrete model to understand and do arithmetic with positive and negative integers

- Applying algebraic ideas, including In-Out tables, in geometric settings
- Developing proofs concerning consecutive sums and other topics

Read "Patterns: A Mathematical Commentary" by Eric Robinson, Professor of Mathematics at Ithaca College, Ithaca, New York.

**Progression**

In *The Importance of Patterns*, the unit opens with an introduction to functions and their representations. Students begin to build their ability to tackle novel mathematical problems and have their first experiences with graphing calculators. The activities in *Communicating About Mathematics* build on the strands begun in *The Importance of Patterns*, focusing on the written and oral communication of mathematical ideas. In *Investigations*, students explore several rich mathematical tasks while employing the tools and techniques they have developed so far. Finally, in *Putting It Together*, students bring all of their new mathematical tools and techniques, as well as their developing identity as a learning community, to bear on a group of summary activities.

The Importance of Patterns

Communicating About Mathematics

Investigations

Putting It Together

Supplemental Activities

Unit Assessments
Introduction

The Overland Trail Unit Overview

Intent
This unit uses the mid-nineteenth century migration of settlers from the eastern part of the United States across the west to California as the context for laying the foundation for algebraic thinking.

Mathematics
Building on students’ work in Patterns, this unit develops the central mathematical idea of functions and their representations. Students will move among the following four “faces” of functions:

- Situations
- Graphs
- Tables
- Rules

The focus of this unit is on linear functions. Students will use starting values and rate of change to characterize linear functions, build In-Out tables, draw graphs, and write equations to represent specific contexts. They will use tables, graphs, and symbols to solve linear equations and systems of linear equations. They will fit lines to real data and use graphs and symbols representing these lines to solve problems in the context of the unit.

The main concepts and skills that students will encounter and practice during the course of this unit can be summarized by category.

Constraints and Decision Making
- Creating examples that fit a set of constraints
- Finding numbers that fit several conditions
- Using tables of information and lines of best fit to make predictions and estimates
- Working with mean and median

Algorithms, Variables, and Notation
- Strengthening understanding of the distributive property
• Developing numeric algorithms for problem situations
• Expressing algorithms in words and symbols
• Interpreting algebraic expressions in words using summary phrases
• Developing meaningful algebraic expressions

**Basics of Graphing**

• Reviewing the coordinate system
• Interpreting graphs intuitively and using graphs intuitively to represent situations
• Making graphs from tabular information
• Quantifying graphs with appropriate scales
• Using graphs to represent two-variable equations and data sets
• Using multiple representations—graphs, tables, and algebraic relationships—to describe situations

**Linear Equations, Graphs, and Situations**

• Finding and interpreting lines of best fit intuitively
• Seeing the role of constant rate in linear situations
• Using rates and starting values, or other data points, to create equations for straight lines
• Laying the groundwork for the concept of slope
• Using the point of intersection of two graphs to find values that satisfies two conditions
• Solving linear equations for one variable in terms of another
• Solving problems involving two linear conditions
• Solving linear equations in one variable

**Graphs and Technology**

• Making and interpreting graphs on a graphing calculator
• Using the zoom and trace features to get information from a graphing calculator

**Progression**

A wagon train trip from Missouri to California in the 1840s drives the unit. After students read about the history of westward migration and “adopt” families who are making the trip along the Overland Trail, they use variables and graphs to represent a variety of situations encountered by these families. They fit lines to data and identify the key features of linear relationships, and they develop graphing
Introduction

Cookies Unit Overview

Intent
The focus of this unit is the use of graphs of linear equations and inequalities to analyze and solve problems. The overarching goal of the unit is for students to deepen their understanding of the relationship between equations or inequalities and their graphs and to reason about and solve problems both symbolically and graphically.

The central unit problem involves a bakery that produces two kinds of cookies. The bakery faces constraints on ingredients, preparation time, and oven space. The two kinds of cookies require different amounts of ingredients and earn different amounts of profit. The task for students is to determine how many of each kind of cookie the bakery should produce to maximize overall profit.

Mathematics
The central mathematical focus of Cookies is the formulation and solution of problems of optimization, or linear programming problems. In problems of this type, a linear function is to be optimized and a set of linear conditions constrains the possible solutions. Linearity is an important feature of these two-variable problems, in two ways:

- The constraints are linear, so the feasible region is a polygon and its vertices can be found by solving pairs of linear equations.
- The expression to be maximized or minimized is linear, so the points that give this expression a particular value lie on a straight line, and investigating a series of values produces a family of parallel lines.

The linear programming problems that students encounter in this unit involve only two variables and a limited number of constraints. Their solutions are therefore easier to understand graphically, and the algebra needed to find their exact solutions is manageable.
The main concepts and skills that students will encounter and practice during the unit are summarized here.

**Using Variables to Represent Problems**
- Expressing and interpreting constraints using inequalities
- Expressing problem situations using systems of linear equations

**Working with Variables, Equations, and Inequalities**
- Finding equivalent equations and inequalities
- Solving linear equations for one variable in terms of another
- Developing and using a method for solving systems of two linear equations in two unknowns
- Recognizing inconsistent systems and dependent systems

**Graphing**
- Graphing linear inequalities and systems of linear inequalities
- Finding the equation of a straight line and the inequality for a half plane
- Using graphing calculators to draw feasible regions
- Relating the intersection point of graphed lines to the common solution of the related equations
- Using graphing calculators to estimate coordinates of points of intersection

**Reasoning Based on Graphs**
- Recognizing that setting a linear expression equal to a series of constants produces a family of parallel lines
- Finding the maximum or minimum of a linear equation over a region
- Examining how the parameters in a problem affect the solution
• Developing methods of solving linear programming problems with two variables

Creating Word Problems

• Creating problems that can be solved using two equations in two unknowns

• Creating problems that can be solved by linear programming methods

Progression

Following an introduction to the unit problem, students explore the graphing of linear inequalities and create graphs of feasible regions. Using graphical analysis, they maximize and minimize profit and cost functions and learn that the best answers for linear programming problems are at the vertices of the feasible regions. This leads to a focus on methods for solving linear systems. The unit concludes with students creating their own linear programming problems.

Cookies and Inequalities

Picturing Cookies

Using the Feasible Region

Points of Intersection

Cookies and the University

Creating Problems

Supplemental Activities

Unit Assessments
Introduction

Fireworks Unit Overview

Intent

This unit uses a variety of contexts—projectile motion, areas and volumes, the Pythagorean theorem, and economics—to develop students’ understanding of quadratics functions and their representations, as well as methods for solving quadratic equations.

The central problem involves a rocket used to launch a fireworks display. The height of the rocket is described by a quadratic function, and the questions involve vertices and $x$-intercepts, which are fundamental features of the graphs of quadratic functions.

Over the course of the unit, students strengthen their abilities to work with algebraic symbols and to relate algebraic representations to problem situations. Specifically, they see that rewriting quadratic expressions in special ways, either in factored form or in vertex form, provides insight into the graphs of the corresponding functions. Establishing this connection between algebra and geometry is a primary goal of the unit.

Mathematics

Fireworks focuses on the use of quadratic functions to represent a variety of real-world situations and on the development of algebraic skills for working with those functions. Experiences with graphs play an important role in understanding the behavior of quadratic functions.

The main concepts and skills students will encounter and practice during the unit are summarized here.

Mathematical Modeling

- Expressing real-world situations in terms of functions and equations
- Applying mathematical tools to models of real-world problems
- Interpreting mathematical results in terms of real-world situations

Graphs of Quadratic Functions

- Understanding the roles of the vertex and $x$-intercept in the graphs of quadratic functions
- Recognizing the significance of the sign of the $x^2$ term in determining the orientation of the graph of a quadratic function
- Using graphs to understand and solve problems involving quadratic functions

Working with Algebraic Expressions

- Using an area model to understand multiplication of binomials, factoring of quadratic expressions, and completing the square of quadratic expressions
• Transforming quadratic expressions into vertex form
• Simplifying expressions involving parentheses
• Identifying certain quadratic expressions as perfect squares

Solving Quadratic Equations
• Interpreting quadratic equations in terms of graphs and vice versa
• Estimating x-intercepts using a graph
• Finding roots of an equation using the vertex form of the corresponding function
• Using the zero product rule of multiplication to solve equations by factoring

Progression

The unit begins with a graphical treatment of quadratic functions. The area model for multiplication is then used as the basis for the development of a set of manipulative skills. Students use these skills and graphs to solve problems drawn from a variety of real-world contexts. In each case, the interplay between symbolic and graphical representations of quadratic functions is emphasized. In addition, there are two POWs in the unit and the option of including a third from the list of supplemental activities.

A Quadratic Rocket
The Form of It All
Putting Quadratics to Use
Back to Bayside High
Intercepts and Factoring
Supplemental Activities
Unit Assessments
Introduction

All About Alice Unit Overview

Intent

This unit uses Lewis Carroll's story Alice's Adventures in Wonderland as the context in which students define the exponential function, derive properties of exponents, and use exponents to solve problems.

Mathematics

Unlike most other IMP units, All About Alice has no central problem to solve. Instead, there is a general context to the unit, as in the Year 1 unit The Overland Trail.

In particular, the Alice story provides a metaphor for understanding exponents. When Alice eats an ounce of cake, her height is multiplied by a particular whole-number amount; when she drinks an ounce of beverage, her height is multiplied by a particular fractional amount. Using this metaphor, students reason about exponential growth and decay.

Students use several approaches to extend exponentiation beyond positive integers: a contextual situation, algebraic laws, graphs, and number patterns. They then apply principles of exponents to study logarithms and scientific notation.

The main concepts and skills students will encounter and practice during the course of this unit are summarized by category here.

Extending the Operation of Exponentiation

Defining the operation for an exponent of zero
Defining the operation for negative integer exponents
Defining the operation for fractional exponents

Laws of Exponents

Developing the additive law of exponents
Developing the law of repeated exponentiation

Graphing

Describing the graphs of exponential functions
Comparing graphs of exponential functions for different bases
Describing the graphs of logarithmic functions
Comparing graphs of logarithmic functions for different bases
Logarithms
Understanding the meaning of logarithms
Making connections between exponential and logarithmic equations

Scientific Notation
Converting numbers from ordinary notation to scientific notation, and vice versa
Developing principles for doing computations using scientific notation
Using the concept of order of magnitude in estimation

Progression
The unit begins with a brief introduction to the Alice metaphor. Next, students
develop a set of rules for computing with exponents and generalize these rules to
include zero, negative, and fractional exponents. Finally, students "undo" the
exponential function to define the logarithmic function and learn about scientific
notation. There are two POWs in this unit.

Who's Alice?
Extending Exponentiation
Curiouser and Curiouser!
Turning Exponents Around
Supplemental Activities
Unit Assessments
Meadows or Malls?

Intent
Students solve a linear programming problem in six variables, using matrices to solve systems of linear equations.

Mathematics
The main concepts and skills that students will encounter and practice during this unit are:

General Linear Programming
- Seeing that for two-variable problems, the optimal value always occurs at a corner point of the feasible region
- Generalizing the corner-point principle to more than two variables
- Recognizing that for two-variable problems, corner points can be found as the intersections of lines corresponding to constraint equations or inequalities
- Generalizing the method of finding corner points to more than two variables

Solving Linear Equations
- Using substitution, graphing, and guess-and-check methods to solve systems of linear equations in two variables
- Developing and using the elimination method to solve systems of linear equations in two or more variables
- Using the concepts of inconsistent, dependent, and independent systems of equations

Geometry in the Plane and in 3-Space
- Extending the concept of coordinates to three variables by introducing a third axis perpendicular to the first two
- Graphing linear equations in three variables and recognizing that these graphs are planes in 3-space
- Seeing that two distinct points always determine a unique line and that two distinct lines in the plane determine a unique point unless the lines are parallel
- Examining the possible intersections of planes in 3-space
- Relating the possible intersections of lines and planes to the algebra of solving linear systems in two or three variables
Matrix Algebra
- Using matrices to represent information
- Using problem situations to motivate and develop the definitions of matrix addition and multiplication
- Examining whether matrix operations have certain properties, such as associativity and commutativity

Matrices and Systems of Linear Equations
- Seeing that systems of linear equations are equivalent to certain types of matrix equations
- Recognizing the role of identity and inverse elements in solving certain types of matrix equations
- Finding matrix inverses by hand by solving systems of linear equations
- Understanding the relationship between a system of linear equations having a unique solution and the coefficient matrix being invertible

Technology
- Entering matrices and doing matrix operations on a graphing calculator
- Using matrix inversion on a graphing calculator to solve systems of linear equations

Progression
In Recreation Versus Development: A Complex Problem, the unit opens with a city planning dilemma that is a linear programming problem in six variables. After a brief look at that problem, students spend several days reviewing the use of graphing to solve linear programming problems involving two variables, in A Strategy for Linear Programming. They see the need for a more general approach that does not require graphing when working with six variables in the central unit problem.

Based on work with several sample problems, students create a strategy for solving linear programming problems involving two variables without referring to a graph. In particular, they see that the optimal solution occurs at a corner point of the feasible region and that they can identify corner points by finding the common solution to pairs of linear equations involved in the problem. They test their algorithm (and correct or improve it if necessary) by working on other two-variable problems.

Equations, Points, Lines and Planes involves generalizing these concepts to three dimensions. Students are introduced to the coordinate system for three variables and see that the graph of a linear equation in three variables is a plane. They then investigate the possibilities for intersections of planes in 3-
space. They see that "usually" three such planes determine a single point but that "exceptional" cases can occur, just as two lines in the plane do not necessarily intersect in a single point.

With these geometric insights, students apply their linear programming strategy to a three-variable problem in Cookies, Cookies, Cookies. They see that the feasible region for a three-variable linear programming problem is probably some sort of polyhedron and that the desired corner points are common solutions to sets of three linear equations involved in the problem. Through other problems, they see that if a constraint is an equation rather than an inequality, they should include it in all the combinations of constraints they use in their search for corner points.

As part of their work early in the unit, students review substitution methods for solving pairs of linear equations. Through the development of a general strategy for solving linear programming problems, they see that solving such systems is an essential part of the process, and in Equations, Equations, Equations, they develop the elimination method, which they extend beyond two variables. Equations and More Variables in Programming continues further development of writing constraints (including equations) and refining the process of solving linear programming problems.

Students also see, however, that as the number of variables increases, solving each system becomes more difficult. So they are highly motivated to find a tool that will do some of the work for them. In Saved by the Matrices!, looking at the elimination method leads them to focus on the matrix of coefficients. They develop the operations of addition and multiplication of matrices through problem contexts and learn how to use graphing calculators to do these operations on matrices.

Next, students investigate the process of representing systems of linear equations by matrices. Through an analogy with numerical equations, they develop the concepts of an identity element and an inverse, and they see that inverses of matrices are the key to solving linear systems. They learn that graphing calculators will also find inverses of matrices (when they exist), up to a certain size.

The use of the graphing calculators and matrix methods greatly simplifies the solving of linear programming problems. In Solving Meadows or Malls?, Students end the unit by solving the original six-variable problem.

Recreation Versus Development: A Complex Problem
A Strategy for Linear Programming
Equations, Points, Lines and Planes
Small World, Isn’t It?

Intent
In this unit, students solve a problem involving population growth by fitting a function to a set of data. In preparation for this, they consider the nature of various mathematical descriptions of growth, including linear and exponential functions, slope, and derivatives. Students also learn about common and natural logarithms.

Mathematics
The main concepts and skills that students will encounter and practice during this unit are:

Rate of Change
• Evaluating average rate of change in terms of the coordinates of points on a graph
• Understanding the relationship between the rate of change of a function and the appearance of its graph
• Realizing that in many contexts, the rate of growth or decline with respect to time in a population is proportional to the population

Slope and Linear Functions
• Developing an algebraic definition of slope
• Proving, using similarity, that a line has a constant slope
• Understanding the significance of a negative slope for a graph and an applied context
• Seeing that the slope of a line is equal to the coefficient of x in the $y = a + bx$ representation of the line
• Using slope to develop equations for lines

Derivatives
• Developing the concept of the derivative of a function at a point
• Seeing that the derivative of a function at a point is the slope of the tangent line at that point
• Finding numerical estimates for the derivatives of functions at specific points
• Working with the derivative of a function as a function in itself
• Realizing that for functions of the form $y = bx^c$, the derivative at each point of the graph is proportional to the $y$-value at that point
**Exponential and Logarithmic Functions**

- Using exponential functions to model real-life situations
- Strengthening understanding of logarithms
- Reviewing and applying the principles that $a^b \cdot a^c = a^{b+c}$ and $(a^b)^c = a^{bc}$
- Understanding and using the fact that $a^{\log_a b} = b$
- Discovering that any exponential function can be expressed using any positive number other than 1 as a base
- Learning the meaning of the terms natural logarithm and common logarithm
- Using an exponential function to fit a curve to numerical data

**The Number e and Compound Interest**

- Estimating the value of $b$ for which the function $y = b^x$ has a derivative at each point on its graph equal to the $y$-value at that point
- Developing and using a formula for compound interest
  - Seeing that expressions of the form $\left(1 + \frac{1}{n}\right)^n$ have a limiting value, called $e$, as $n$ increases without bound
  - Learning that the limiting value $e$ is the same number as the special base for exponential functions

Students will work with other concepts in connection with the unit’s Problems of the Week.

**Progression**

In *As the World Grows*, the unit opens with a table of world population data over the past several centuries and asks this question:

*If population growth continues according to its current pattern, how long will it be until people are squashed up against one another?*

To answer this question, students begin with a variety of problems concerning rates of growth, focusing on the idea of an average rate of change, in *Average Growth*.

In *All In a Row*, the unit then focuses specifically on the case of constant change, represented by linear functions. Students develop the concept of slope, seeing that the rate of change can be represented by a ratio of coordinate differences. They then apply the concept of slope to develop equations for linear functions, based either on the slope and a point or on two points. They see that the slope of a straight line is equal to the
coefficient of $x$ in the algebraic function representation of the line. Students also use the concept of similarity to see why the slope of a straight line does not depend on the particular points used to compute the slope.

After this work with linear functions, in Beyond Linearity students examine nonlinear functions, looking for an analogous concept. Working with the notion of instantaneous velocity, they develop the concept of a derivative. This crucial concept is examined from several perspectives:

- **In terms of real-life situations:** Through examples such as the speed of a falling object, students connect the abstract concept of a derivative with more concrete models.
- **Numerically:** Students estimate derivatives numerically by computing the slope of the segment connecting "close" points on the graph.
- **Graphically:** Using a graphing calculator, students see that almost any graph they can create will look like a straight line if they zoom in close enough. This leads to discussion of the line tangent to a graph.

In A Model for Population Growth, students then examine two situations that fit a model of exponential growth—an inflation problem and a problem involving growth of an amoeba-like population. This leads to an examination of the derivatives of exponential functions, and students see that every exponential function has a derivative proportional to its $y$-value. This discovery fits with an intuitive analysis of population growth suggesting that the rate (with respect to time) at which a population is increasing should be proportional to the current population. Thus, exponential growth is a natural model to consider in solving the central unit problem.

In the next segment of the unit, The Best Base, students apply their understanding of bases and exponents to see that any positive number except 1 can be used as the base for any exponential function. Students estimate a value for the special base $b$ for which the function $y = b^x$ has a derivative that is actually equal to its $y$-value. Then they study compound interest, recognizing that the number $e$ that comes out of the compounding problem is the same number as the special base for the exponential function.