ANGLE SUMS ON POLYTOPES AND POLYTOPAL COMPLEXES

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Cornell University 2006

We will study the angle sums of polytopes, working to exploit the analogy between
the \( f \)-vector of faces in each dimension and the \( \alpha \)-vector of angle sums. The Gram
relation on the \( \alpha \)-vector is analogous to the Euler relation on the \( f \)-vector. Similarly, the Perles relations on the angle sums of simplicial polytopes are analogous
to the Dehn-Sommerville relations.

First we describe the spaces spanned by the angle sums of certain classes of
polytopes, as recorded in the \( \alpha \)-vector and the \( \alpha \)-\( f \)-vector. Families of polytopes
are constructed whose angle sums span the spaces of polytopes defined by the
Gram and Perles equations. This shows that the dimension of the affine span of
the space of angle sums of simplices is \( \left\lfloor \frac{d-1}{2} \right\rfloor \), and that of the combined angle sums
and face numbers of simplicial polytopes and general polytopes are \( d-1 \) and \( 2d-3 \),
respectively.

Next we consider angle sums of polytopal complexes. We define the angle char-
acteristic on the \( \alpha \)-vector in analogy to the Euler characteristic. Then we consider
the effect of a gluing operation to construct new complexes on the angle and Eu-
ler characteristics. We show that the changes in the two correspond and that,
in the case of certain odd-dimensional polytopal complexes, the angle characte-
ristic is half the Euler characteristic. In particular, we show that many non-convex
spheres satisfy the Gram relation and handle-bodies of genus \( g \) constructed via
gluings along disks have angle characteristic $1 - g$.

Finally, we consider spherical and hyperbolic polytopes and polytopal complexes. Spherical and hyperbolic analogs of the Gram relation and a spherical analog of the Perles relation are known, and we show the hyperbolic analog of the Perles relations in a number of cases. Proving this relation for simplices of dimension greater than 3 would finish the proof of this result. Also, we show how constructions on spherical and hyperbolic polytopes lead to corresponding changes in the angle characteristic and Euler characteristic. However, the angle characteristic and Euler characteristic do not have the 1:2 ratio that held for Euclidean polytopal complexes.