Paper folding geometry: how origami beat Euclid
Double the altar
Double the altar
Double the cube
Double the cube
Double the cube
Double the cube?
Double the cube?
Double the cube?
No, octuple the cube!
Double the cube

Volume of the cube $\quad = (\text{edge})^3 = 2$

$\Rightarrow \text{edge} = \sqrt[3]{2} \approx 1.25992105...$
Double the cube

Volume of the cube $= (\text{edge})^3 = 2$

$\Rightarrow \text{edge} = \sqrt[3]{2} \approx 1.25992105\ldots$
Double the cube

Volume of the cube \( = (\text{edge})^3 = 2 \)

\[ \Rightarrow \text{edge} = \sqrt[3]{2} \approx 1.25992105... \]
Double the square
Double the square
Double the square
Double the square

1
Double the square
The Elements
of Geometry,
of the most ancient
Philosopher
Euclides
of Megara.

Faithfully (now first) translated into the English tongue by
H. Billingsley, Citizen of London.
Whereunto are annexed certaine
Scholias, Demonstrations, and Inventions
of the late Mathematicians, both of some past, and
in this our age.

With a very fruitful Preface made by M. I. Dee,
Specifying the chiefest Mathematical Sciences, what
they are, and wherein commodiously they are
disclosed, certaine new Secrets Mathematical
and Mechanical, with their use, greatly useful.

Imprinted at London by John Daye.
THE FIRST SIX BOOKS OF
THE ELEMENTS OF EUCLID
IN WHICH COLOURED DIAGRAMS AND SYMBOLS
ARE USED INSTEAD OF LETTERS FOR THE
GREATER EASE OF LEARNERS

BY OLIVER BYRNE
SURVEYOR OF HER MAJESTY'S SETTLEMENTS IN THE FALKLAND ISLANDS
AND AUTHOR OF NUMEROUS MATHEMATICAL WORKS

LONDON
WILLIAM PICKERING
1847
THE ELEMENTS OF EUCLID.

BOOK I.

DEFINITIONS.

I.
A *point* is that which has no parts.

II.
A *line* is length without breadth.

III.
The extremities of a line are points.

IV.
A *straight* or right line is that which lies evenly between its extremities.

V.
A *surface* is that which has length and breadth only.

VI.
The extremities of a surface are lines.

VII.
A *plane surface* is that which lies evenly between its extremities.

VIII.
A *plane angle* is the inclination of two lines to one another, in a plane, which meet together, but are not in the same direction.

IX.
A *plane rectilinear angle* is the inclination of two straight lines to one another, which meet together, but are not in the same straight line.
Postulates:

1. Let it be granted that a straight line may be drawn from any one point to any other point.
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2. Let it be granted that a finite straight line may be produced to any length in a straight line.
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1. Let it be granted that a straight line may be drawn from any one point to any other point.
2. Let it be granted that a finite straight line may be produced to any length in a straight line.
3. Let it be granted that a circle may be described with any center at any distance from that center.
Postulates:

1. Let it be granted that a straight line may be drawn from any one point to any other point.
2. Let it be granted that a finite straight line may be produced to any length in a straight line.
3. Let it be granted that a circle may be described with any center at any distance from that center.
4. All right angles are equal.
Postulates:

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2. Let it be granted that a finite straight line may be produced to any length in a straight line.
3. Let it be granted that a circle may be described with any center at any distance from that center.
4. All right angles are equal.
5. If two straight lines meet a third straight line so as to make the two interior angles on the same side less than two right angles, then those two lines will meet if extended indefinitely on the side on which the angles are less than two right angles.
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4. All right angles are equal.
5. Given a straight line and a point not on that line, there exists exactly one line through that point that is parallel to the given line.
Postulates:
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4. All right angles are equal.
5. Given a straight line and a point not on that line, there exists exactly one line through that point that is parallel to the given line.
Proposition 1:
To construct an equilateral triangle on a given finite straight line.
Proposition 1:
To construct an equilateral triangle on a given finite straight line.
Proposition 1:
To construct an equilateral triangle on a given finite straight line.
Proposition 1:
To construct an equilateral triangle on a given finite straight line.
Proposition 9:
To bisect a given rectilinear angle.
Proposition 9:
To bisect a given rectilinear angle.
Proposition 9:
To bisect a given rectilinear angle.
Proposition 9:
To bisect a given rectilinear angle.
Proposition 9:
To bisect a given rectilinear angle.
Proposition 10:
To bisect a given finite straight line.
Proposition 10:
To bisect a given finite straight line.
Proposition 10:
To bisect a given finite straight line.
In a right angled triangle the square on the hypotenuse is equal to the sum of the squares of the sides, and 

On , and 
describe squares, (pr. 46.)

Draw || (pr. 31.)
also draw and .

\[ \frac{c}{a} = \frac{\sqrt{a}}{a} \]

To each add \[ \frac{c}{a} = \frac{\sqrt{a}}{a} \]
and \[ \frac{c}{a} = \frac{\sqrt{a}}{a} \];

\[ \therefore \]

Again, because ||
In a right angled triangle, the square on the hypotenuse is equal to the sum of the squares of the sides, (and).

On and describe squares, (pr. 46.)

Draw || (pr. 31.)
also draw and .

= ,

To each add .

= and = ;

. = .

Again, because || 

In the same manner it may be shown that .

hence .

Q.E.D.
Double the cube

Volume of the cube  = (edge)$^3$ = 2

⇒ edge = $\sqrt[3]{2} \approx 1.25992105...$
What points can we construct?
What points can we construct?
What points can we construct?
What points can we construct?
What points can we construct?
What points can we construct?
What points can we construct?
What numbers can we construct?
What numbers can we construct?
What numbers can we construct?
What numbers can we construct?
What numbers can we construct?
What numbers can we construct?
What numbers can we construct?
What numbers can we construct?
What numbers can we construct?
What numbers can we construct?
What numbers can we construct?

All integers and all fractions with denominator $2^k$. 
What numbers can we construct?

All integers and all fractions with denominator $2^k$.
And what else?
Thales’ theorem (intercept theorem):
Thales’ theorem (intercept theorem):
Thales’ theorem (intercept theorem):
Thales’ theorem (intercept theorem):
Thales’ theorem (intercept theorem):

\[ \frac{b}{a} = \frac{d}{c} \]
A = B = height of Thales
A = B = height of Thales

\[ C = 32m + \frac{230m}{2} = 147m \]
A = B = height of Thales

\[ C = 32m + \frac{230m}{2} = 147m \]

\[
\frac{D}{A} = \frac{C}{B}
\]
\[ A = B = \text{height of Thales} \]

\[ C = 32m + \frac{230m}{2} = 147m \]

\[
\begin{align*}
\frac{D}{A} &= \frac{C}{B} \\
D &= 147m
\end{align*}
\]
Thales’ theorem (intercept theorem):

\[
\frac{b}{a} = \frac{d}{c}
\]
\[
\frac{b}{a} = \frac{?}{1}
\]
\[
\frac{b}{a} = \frac{?}{1}
\]
We can construct all fractions!
We can construct all fractions!

\[ \frac{b}{a} = \frac{?}{1} \]
\[ \frac{?}{a} = \frac{d}{1} \]
\[ \frac{?}{a} = \frac{d}{1} \]
$a \times d = \frac{d}{1}$
We can multiply, divide, add and subtract.

\[
\frac{d}{a} = \frac{d}{1}
\]
We can multiply, divide, add and subtract.

\[
\frac{a}{a} = \frac{d}{1}
\]

And what else?
All square roots!
All square roots!
All square roots!
All square roots!

\[ 1^2 + \sqrt{2}^2 = ?^2 \]
All square roots!

\[ 1^2 + \sqrt{2}^2 = ?^2 \]
All square roots!

\[ 1^2 + \sqrt{2}^2 = ?^2 \]
All square roots!

$1^2 + \sqrt{2}^2 = ?^2$

$1^2 + \sqrt{3}^2 = ?^2$
All square roots!

\[
\begin{align*}
1^2 + \sqrt{2}^2 &= ?^2 \\
1^2 + \sqrt{3}^2 &= ?^2
\end{align*}
\]
All square roots!
All square roots!
All square roots!

Ver no quadro que produz todas as $\sqrt{k}$

Espiral de Teodoro (até $\sqrt{17}$). Em 1958, E. Teuffel provou que duas hipotenusas da espiral nunca colidem.
Ver no quadro que produz todas as $\sqrt{k}$.

Espiral de Teodoro (até $\sqrt{17}$). Em 1958, E. Teuffel provou que duas hipotenusas da espiral nunca colidem.
We can construct
We can construct $\sqrt{2}$
We can construct $\sqrt{2}$

$\sqrt{3}$
We can construct $\sqrt{2}$
$\sqrt{3}$

$\sqrt{53}$
We can construct $\sqrt{2}$

$\sqrt{3}$

$\sqrt{53}$

$6\sqrt{7}$
We can construct $\sqrt{2}$
\[
\sqrt{3}
\]

\[
6\sqrt{7}
\]
\[
1 + 6\sqrt{7}
\]
\[
\sqrt{53}
\]
We can construct $\sqrt{2}$

$\sqrt{3}$

$\sqrt{53}$

$\frac{2}{3} + 5\sqrt{11}$
We can construct $\sqrt{2}$

$\sqrt{3}$

$\frac{4 - 2\sqrt{15}}{22}$

$6\sqrt{7}$

$1 + 6\sqrt{7}$

$\sqrt{53}$

$\frac{2}{3} + 5\sqrt{11}$
We can construct \( \sqrt{2} \), \( \sqrt{3} \), \( \frac{4-2\sqrt{15}}{22} \), \( -\frac{9}{13} + \frac{1}{37} \sqrt{10} \), \( 6\sqrt{7} \), \( 1 + 6\sqrt{7} \), \( \sqrt{53} \), and \( \frac{2}{3} + 5\sqrt{11} \).
We can construct $\sqrt{2}$

$\frac{4-2\sqrt{15}}{22}$

$\frac{5}{6}\sqrt{3} - 2\sqrt{\frac{10}{7}}$

$\sqrt{53}$

$\frac{2}{3} + 5\sqrt{11}$

$-\frac{9}{13} + \frac{1}{37}\sqrt{10}$

$6\sqrt{7}$

$1 + 6\sqrt{7}$
We can construct \( \sqrt{2} \) and \( \sqrt{3} \):

\[
\frac{4-2\sqrt{15}}{22}, \quad 6\sqrt{7}, \quad 1+6\sqrt{7}, \quad \frac{5}{6}\sqrt{3} - 2\sqrt{\frac{10}{7}}, \quad \sqrt{53}
\]

\[
-\frac{9}{13} + \frac{1}{37} \sqrt{10}, \quad \frac{8}{\sqrt{5}}, \quad \frac{2}{3} + 5\sqrt{11}
\]
We can construct $\sqrt{2}$

$\sqrt{3}

\frac{4-2\sqrt{15}}{22}$

$6\sqrt{7}$

$1 + 6\sqrt{7}$

$\frac{5}{6}\sqrt{3} - 2\sqrt{\frac{10}{7}}$

$\sqrt{53}$

$\frac{\sqrt{2}}{\sqrt{3}}$

$-\frac{9}{13} + \frac{1}{37}\sqrt{10}$

$\frac{8}{\sqrt{5}}$

$\frac{2}{3} + 5\sqrt{11}$
We can construct \( \sqrt{2} \)
\( \sqrt{3} \)
\( -\, \frac{9}{13} + \frac{1}{37} \sqrt{10} \)
\( \frac{4-2\sqrt{15}}{22} \)
\( 6\sqrt{7} \)
\( 1 + 6\sqrt{7} \)
\( \frac{8}{\sqrt{5}} \)
\( \sqrt{\frac{23}{41}} \)
\( \frac{5}{6} \sqrt{3} - 2\sqrt{\frac{10}{7}} \)
\( \sqrt{53} \)
\( \frac{\sqrt{2}}{\sqrt{3}} \)
\( \frac{2}{3} + 5\sqrt{11} \)
We can construct \( \sqrt{2} \) and \( \sqrt{3} \):

\[
\begin{align*}
\frac{4-2\sqrt{15}}{22} & \quad 6\sqrt{7} \\
\frac{5}{6}\sqrt{3} - 2\sqrt{\frac{10}{7}} & \quad 1 + 6\sqrt{7} \\
\frac{\sqrt{2}}{\sqrt{3}} & \quad \sqrt{\frac{23}{41}} \\
-4 + 12\sqrt{\frac{19}{92}} & \quad \frac{2}{3} + 5\sqrt{11} \\
-\frac{9}{13} + \frac{1}{37}\sqrt{10} & \quad \frac{8}{\sqrt{5}}
\end{align*}
\]
We can construct $\sqrt{2}$ $\frac{2}{3}$ $\sqrt{3}$ $\frac{4-2\sqrt{15}}{22}$ $\frac{3-\sqrt{22}}{-22+\frac{4}{5}\sqrt{67}}$ $\frac{5}{6}\sqrt{3} - 2\sqrt{\frac{10}{7}}$ $\frac{\sqrt{2}}{\sqrt{3}}$ $\sqrt{53}$ $-4 + 12\sqrt{\frac{19}{92}}$ $-\frac{9}{13} + \frac{1}{37}\sqrt{10}$ $6\sqrt{7}$ $1 + 6\sqrt{7}$ $\frac{8}{\sqrt{5}}$ $\sqrt{\frac{23}{41}}$ $\frac{2}{3} + 5\sqrt{11}$
We can construct $\sqrt{2}$
$\sqrt{3}$
$\frac{4 - 2\sqrt{15}}{22}$
$\frac{3 - \sqrt{22}}{-22 + \frac{4}{5} \sqrt{67}}$
$\frac{5}{6} \sqrt{3} - 2 \sqrt{\frac{10}{7}}$
$\sqrt{53}$
$\frac{\sqrt{2}}{\sqrt{3}}$
$-4 + 12 \sqrt{\frac{19}{92}}$
$-\frac{9}{13} + \frac{1}{37} \sqrt{10}$
$6\sqrt{7}$
$1 + 6\sqrt{7}$
$\sqrt{\frac{23}{41}}$
$\frac{8}{\sqrt{5}}$
$\frac{2}{3} + 5 \sqrt{11}$
$\sqrt{\frac{21}{100}} - \frac{52}{5} \sqrt{26} + \frac{43}{\sqrt{7}} + \frac{38}{83} \sqrt{99}$
We can construct $\sqrt{2}$, $\sqrt{3}$

- $\frac{4-2\sqrt{15}}{22}$
- $\frac{3-\sqrt{22}}{-22+\frac{4}{5}\sqrt{67}}$
- $\frac{5}{6}\sqrt{3} - 2\sqrt{\frac{10}{7}}$
- $\sqrt{\frac{23}{41}}$
- $\frac{\sqrt{2}}{\sqrt{3}}$
- $\sqrt{53}$
- $\frac{\sqrt{2}}{\sqrt{3}}$
- $\sqrt{\frac{21}{100}} - \frac{52}{5}\sqrt{26} + \frac{43}{\sqrt{7}} + \frac{38}{83}\sqrt{99}$

$\frac{1}{37}\sqrt{10}$

$+ \frac{9}{13}$

$1 + 6\sqrt{7}$

$6\sqrt{7}$

etc. $\frac{8}{\sqrt{5}}$

$\frac{2}{3} + 5\sqrt{11}$

$-4 + 12\sqrt{\frac{19}{92}}$
We can construct \( \sqrt{2} \)\( \sqrt{3} \)

\[
\frac{4 - 2\sqrt{15}}{22} \\
\frac{3 - \sqrt{22}}{-22 + \frac{4}{5}\sqrt{67}}
\]

\[
\frac{\sqrt{3}}{6} - 2\sqrt{\frac{10}{7}} \\
\sqrt{53}
\]

\[
\frac{\sqrt{2}}{\sqrt{3}} \\
-4 + 12\sqrt{\frac{19}{92}}
\]

And what about \( \sqrt{1 + 3\sqrt{2}} \)?

\[
-\frac{9}{13} + \frac{1}{37}\sqrt{10} \\
6\sqrt{7} \\
1 + 6\sqrt{7}
\]

\[
\sqrt{\frac{23}{41}} \\
\sqrt{\frac{21}{100}} - \frac{52}{5}\sqrt{26} + \frac{43}{\sqrt{7}} + \frac{38}{83}\sqrt{99}
\]

etc.

\[
\frac{8}{\sqrt{5}}
\]

\[
\frac{2}{3} + 5\sqrt{11}
\]

Sunday, March 3, 13
All square roots?

\[
\frac{a}{?} = \frac{?}{1}
\]
All square roots?

\[
\frac{a}{?} = \frac{?}{1}
\]

? = \sqrt{a}
All square roots!
All square roots!
All square roots!
All square roots!
All square roots!
We can construct $\sqrt{2}$

$\sqrt{3}$

$-\frac{9}{13} + \frac{1}{37} \sqrt{10}$

$\frac{4-2\sqrt{15}}{22}$

$\frac{3-\sqrt{22}}{-22+\frac{4}{5} \sqrt{67}}$

$\frac{5}{6} \sqrt{3} - 2 \sqrt{\frac{10}{7}}$

$\frac{6\sqrt{7}}{1 + 6\sqrt{7}}$

$\sqrt{\frac{23}{41}}$

etc. $\frac{8}{\sqrt{5}}$

$\sqrt{53}$

$\frac{\sqrt{2}}{\sqrt{3}}$

$-4 + 12\sqrt{\frac{19}{92}}$

$\frac{2}{3} + 5\sqrt{11}$

$\sqrt{\frac{21}{100}} - \frac{52}{5} \sqrt{26} + \frac{43}{\sqrt{7}} + \frac{38}{83} \sqrt{99}$
We can construct $\sqrt{2}$

$\sqrt{3}$

$\frac{4-2\sqrt{15}}{22}$

$\frac{3-\sqrt{22}}{-22+\frac{4}{5}\sqrt{67}}$

$\sqrt{\frac{23}{41}}$

$\sqrt{53}$

$\sqrt{2} / \sqrt{3}$

$\sqrt[3]{3} - 2\sqrt{\frac{10}{7}}$

$\sqrt{\frac{19}{92}}$

$\frac{2}{3} + 5\sqrt{11}$

$\frac{6\sqrt{7}}{1+6\sqrt{7}}$

$\frac{-9}{13} + \frac{1}{37}\sqrt{10}$

$\sqrt[1+3\sqrt{2}]{\text{etc.}}$
We can construct $\sqrt{2}$

\[\sqrt{3}\]

\[2 + \sqrt{67} - 3\frac{59}{\sqrt{12}}\]

\[\frac{4 - 2\sqrt{15}}{22}\]

\[-\frac{9}{13} + \frac{1}{37}\sqrt{10}\]

\[6\sqrt{7}\]

\[1 + 6\sqrt{7}\]

\[\frac{5}{6}\sqrt{3} - 2\sqrt{\frac{10}{7}}\]

\[-22 + \frac{4}{5}\sqrt{67}\]

\[\sqrt{\frac{23}{41}}\]

\[\sqrt{53}\]

\[\frac{\sqrt{2}}{\sqrt{3}}\]

\[-4 + 12\sqrt{\frac{19}{92}}\]

\[\frac{2}{3} + 5\sqrt{11}\]

\[\sqrt{\frac{21}{100}} - \frac{52}{5}\sqrt{26} + \frac{43}{\sqrt{7}} + \frac{38}{83}\sqrt{99}\]

etc. \[\frac{8}{\sqrt{5}}\]
We can construct \( \sqrt{2} \) and \( \sqrt{3} \)

\[
\frac{4 - 2\sqrt{15}}{22}
\]

\[
\frac{3 - \sqrt{22}}{-22 + \frac{4}{5}\sqrt{67}}
\]

\[
\frac{5}{6}\sqrt{3} - 2\sqrt{\frac{10}{7}}
\]

\[
\frac{\sqrt{2}}{\sqrt{3}}
\]

\[
-4 + 12\sqrt{\frac{19}{92}}
\]

\[
\sqrt{1 + 3\sqrt{2}}
\]

\[
-\frac{9}{13} + \frac{1}{37}\sqrt{10}
\]

\[
2 + \sqrt{67 - 3\frac{59}{\sqrt{12}}}
\]

\[
6\sqrt{7}
\]

\[
1 + 6\sqrt{7}
\]

\[
\sqrt{\frac{5 + \sqrt{7}}{7 + \sqrt{5}}}
\]

\[
\sqrt{\frac{23}{41}}
\]

\[
\sqrt{53}
\]

\[
\frac{8}{\sqrt{5}}
\]

\[
\sqrt{\frac{21}{100}} - \frac{52}{5}\sqrt{26} + \frac{43}{\sqrt{7}} + \frac{38}{83}\sqrt{99}
\]

etc.
We can construct $\sqrt{2}$

$$\frac{4 - 2\sqrt{15}}{22}$$

$$\frac{3 - \sqrt{22}}{-22 + \frac{4}{5} \sqrt{67}}$$

$$\frac{5}{6} \sqrt{3} - 2 \sqrt{\frac{10}{7}}$$

$$\frac{\sqrt{2}}{\sqrt{3}}$$

$$-4 + 12 \sqrt{\frac{19}{92}}$$

$$\sqrt{53}$$

sqrt $1 + 3\sqrt{2}$

$$\sqrt{\frac{23}{41}}$$

$$\sqrt{\frac{\sqrt{345}}{7} - 2\sqrt{91} - \sqrt{\sqrt{5}} + \frac{33}{\sqrt{2 + \sqrt{3}}}$$

$$\frac{2}{3} + 5\sqrt{11}$$

$$\sqrt{\frac{21}{100}} - \frac{52}{5} \sqrt{26} + \frac{43}{\sqrt{7}} + \frac{38}{83} \sqrt{99}$$
We can construct $\sqrt{2}$ \\
$\sqrt{3}$ \\
$\sqrt{47}$ \\
$\frac{4-2\sqrt{15}}{22}$ \\
$\frac{5}{6} \sqrt{3} - 2 \sqrt{\frac{10}{7}}$ \\
$\frac{\sqrt{2}}{\sqrt{3}}$ \\
$\sqrt{53}$ \\
$-4 + 12 \sqrt{\frac{19}{92}}$ \\
$\sqrt{1 + 3\sqrt{2}}$ \\
$\sqrt{\frac{23}{41}}$ \\
$\sqrt{\frac{\sqrt{345}}{7}} - 2\sqrt{91} - \sqrt{\sqrt{5}} + \frac{33}{\sqrt{2+\sqrt{3}}}$ \\
$\frac{2}{3} + 5\sqrt{11}$ \\
$\frac{\sqrt{\frac{21}{100}}}{20} - \frac{52}{5} \sqrt{26} + \frac{43}{\sqrt{7}} + \frac{38}{83} \sqrt{99}$ \\
$-\frac{9}{13} + \frac{1}{37} \sqrt{10}$ \\
$1 + 6\sqrt{7}$ \\
$\sqrt{\frac{5+\sqrt{7}}{7+\sqrt{5}}}$ \\
e tc. \qquad \frac{8}{\sqrt{5}}$
We can construct \( \sqrt{2} \), \( \sqrt{3} \), \( \sqrt{1 + 3\sqrt{2}} \), \( -\frac{9}{13} + \frac{1}{37}\sqrt{10} \), \( 2 + \sqrt{67 - 3\frac{59}{\sqrt{12}}} \), \( 6\sqrt{7} \), \( \sqrt{\frac{5 + \sqrt{7}}{7 + \sqrt{5}}} \), \( \frac{8}{\sqrt{5}} \), \( \sqrt{\frac{23}{41}} \), etc.

\[
\begin{align*}
\sqrt{53} &= \sqrt{\frac{\sqrt{345}}{7} - 2\sqrt{91} - \sqrt{\sqrt{5}} + \frac{33}{\sqrt{2} + \sqrt{3}}} \\
&= \sqrt{\frac{21}{100}} - \frac{52}{5}\sqrt{26} + \frac{43}{\sqrt{7}} + \frac{38}{83}\sqrt{99}
\end{align*}
\]
We can construct \( \sqrt{2} \) 

\[
\frac{8\sqrt{47}}{22} - 2 + \sqrt{67 - 3 \frac{59}{\sqrt{12}}}
\]

\[
\frac{5}{6} \sqrt{3} - 2 \sqrt{\frac{10}{7}}
\]

\[
\frac{\sqrt{2}}{\sqrt{3}}
\]

\[
-4 + 12 \sqrt{\frac{19}{92}}
\]

And what else?

\[
\sqrt{1 + 3\sqrt{2}}
\]

\[
-\frac{9}{13} + \frac{1}{37} \sqrt{10}
\]

\[
6\sqrt{7} + 1 + 6\sqrt{7}
\]

\[
\sqrt{\frac{5 + \sqrt{7}}{7 + \sqrt{5}}}
\]

\[
\sqrt{\frac{23}{41}}
\]

\[
\sqrt{\frac{\sqrt{345}}{7} - 2\sqrt{91} - \sqrt{\sqrt{5}} + \frac{33}{\sqrt{2 + \sqrt{3}}}}
\]

\[
\frac{2}{3} + 5\sqrt{11}
\]

\[
\sqrt{\frac{21}{100}} - \frac{52}{5} \sqrt{26} + \frac{43}{\sqrt{7}} + \frac{38}{83} \sqrt{99}
\]
We can construct $\sqrt{2}$

$\sqrt{3}$

$\sqrt{47}$

$2 + \sqrt{67 - 3 \frac{59}{\sqrt{12}}}$

$6\sqrt{7}$

$1 + 6\sqrt{7}$

$\sqrt{\frac{5 + \sqrt{7}}{7 + \sqrt{5}}}$

$\sqrt{\frac{23}{41}}$

$\sqrt{53}$

$\sqrt{\frac{345}{7}} - 2\sqrt{91} - \sqrt{\sqrt{5}} + \frac{33}{\sqrt{2 + \sqrt{3}}}$

$\frac{2}{3} + 5\sqrt{11}$

$-\frac{9}{13} + \frac{1}{37} \sqrt{10}$

$\sqrt{\frac{21}{100}} - \frac{52}{5} \sqrt{26} + \frac{43}{\sqrt{7}} + \frac{38}{83} \sqrt{99}$

And what else? Nothing else! 

Sunday, March 3, 13
In general, it’s impossible to trisect an angle \( \theta \) with ruler and compass.
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In general, the number $\cos \frac{\theta}{3}$ is not constructible with ruler and compass.
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$$\cos \theta = 4(\cos \frac{\theta}{3})^3 - 3(\cos \frac{\theta}{3})$$
In general, it’s impossible to trisect an angle $\theta$ with ruler and compass.

In general, the number $\cos \frac{\theta}{3}$ is not constructible with ruler and compass.

$$\cos \theta = 4(\cos \frac{\theta}{3})^3 - 3(\cos \frac{\theta}{3}) \quad \Longrightarrow \quad \frac{1}{2} = 4x^3 - 3x$$
Given a circle, build a square with the same area.
Given a circle, build a square with the same area.
Given a circle, build a square with the same area.

area of the circle $= \pi r^2$
Given a circle, build a square with the same area.

area of the circle = $\pi r^2$

area of the square = $(\text{edge})^2$
Given a circle, build a square with the same area.

area of the circle = $\pi r^2$
area of the square = $(\text{edge})^2$
edge = $\sqrt{\pi r}$
Given a circle, build a square with the same area.

area of the circle = $\pi r^2$

area of the square = $(\text{edge})^2$

dedge = $\sqrt{\pi} \ r$

Is $\sqrt{\pi}$ constructible with ruler and compass?
Given a circle, build a square with the same area.

area of the circle = $\pi r^2$
area of the square = $(\text{edge})^2$
edge = $\sqrt{\pi r}$

Is $\pi$ constructible with ruler and compass?
Given a circle, build a square with the same area.

area of the circle = $\pi r^2$

area of the square = $(\text{edge})^2$

$\text{edge} = \sqrt{\pi r}$

Is $\pi$ constructible with ruler and compass? No.
$n = 2^k \times \text{product of distinct Fermat primes}$
\[ n = 2^k \times \text{product of distinct Fermat primes} \]

\[ F_n = 2^{2^n} + 1 \]
\( n = 2^k \times \text{product of distinct Fermat primes} \)

\[ F_0 = 3, \ F_1 = 5, \ F_2 = 17, \ F_3 = 257, \ F_4 = 65537, \ F_5 \text{ is not prime,} \ldots \]
Dividing into thirds

Darren Scott
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Sunday, March 3, 13
Dividing into thirds

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2/3

1/3

2

3

1

3

2

3

2/3

1/3

Sunday, March 3, 13
Dividing into thirds

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Sunday, March 3, 13
Axiom 1:
Axiom 1:
Axiom 1:

Axiom 2:
Axiom 1:

Axiom 2:
Axiom 1:

Axiom 2:

Axiom 3:
Axiom 1:

Axiom 2:

Axiom 3:

Axiom 4:
Peter Messer’s construction of $\sqrt[3]{2}$:
Peter Messer’s construction of $\sqrt[3]{2}$:
Peter Messer’s construction of $\sqrt[3]{2}$:

$$BP = \sqrt[3]{2} \ AP$$
Angle trisection:
Angle trisection:
Angle trisection:

\[
\hat{B}AD = \frac{1}{3} \hat{BAC}
\]
Angle trisection:

\[ \hat{BAD} = \frac{1}{3} \hat{BAC} \]
Angle trisection:

\[ \hat{BAD} = \frac{1}{3} \hat{BAC} \]
Angle trisection:

$$\hat{BAD} = \frac{1}{3} \hat{BAC}$$
And...?
And...?
And...?

No.
$n = 2^k \times \text{product of distinct Fermat primes}$

$F_n = 2^{2^n} + 1$

$F_0 = 3, F_1 = 5, F_2 = 17, F_3 = 257, F_4 = 65537, F_5$ is not prime, ...
\[ n = 2^k \times \text{product of distinct Fermat primes} \]

\[ F_n = 2^{2^n} + 1 \]

\[ F_0 = 3, \ F_1 = 5, \ F_2 = 17, \ F_3 = 257, \ F_4 = 65537, \ F_5 \text{ is not prime, } \ldots \]
$n = 2^k \times \text{product of distinct Fermat primes}$

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$F_0 = 3, F_1 = 5, F_2 = 17, F_3 = 257, F_4 = 65537, F_5$ is not prime, ...
\[ F_n = 2^{2^n} + 1 \]

\[ F_0 = 3, \quad F_1 = 5, \quad F_2 = 17, \quad F_3 = 257, \quad F_4 = 65537, \quad F_5 \text{ is not prime}, \ldots \]
\[n = 2^k 3^l \times \text{product of distinct Pierpont primes}\]

\[2^u 3^v + 1\]

\[F_0 = 3, \ F_1 = 5, \ F_2 = 17, \ F_3 = 257, \ F_4 = 65537, \ F_5 \text{ is not prime, } \ldots\]
\[ n = 2^k 3^l \times \text{product of distinct Pierpont primes} \]

\[ 2^u 3^v + 1 \]

2, 3, 5, 7, 13, 17, 19, 37, 73, 97, 109, 163, 193, 257, \ldots
Move Point A along the line.
Parabola = \{\text{points at the same distance from the point } O \text{ and the line } l\}
Axiom 6:
directrix: \( y = -1/2 \)
focus: \((0, 1/2)\)

\[ x^2 = 2y \]

\[ y^2 = -4x \]
focus: \((-1, 0)\)
directrix: \(x = 1\)
$y^2 = -4x$

focus: $(-1, 0)$

directrix: $x = 1$

directrix: $y = -1/2$

focus: $(0, 1/2)$
\[ x^2 = 2y \quad \leadsto \quad 2x = 2 \frac{dy}{dx} \]

\[ y^2 = -4x \quad \leadsto \quad 2y \frac{dy}{dx} = -4 \]
\[ x^2 = 2y \implies 2x = 2 \frac{dy}{dx} \implies x = m \]

\[ y^2 = -4x \implies 2y \frac{dy}{dx} = -4 \implies y = \frac{-2}{m} \]
\[ x^2 = 2y \quad \leadsto \quad 2x = 2 \frac{dy}{dx} \quad \leadsto \quad x = m \quad \leadsto \quad (x, y) = (m, \frac{m^2}{2}) \]

\[ y^2 = -4x \quad \leadsto \quad 2y \frac{dy}{dx} = -4 \quad \leadsto \quad y = \frac{-2}{m} \quad \leadsto \quad (x, y) = \left( \frac{-1}{m^2}, \frac{-2}{m} \right) \]
\[ x^2 = 2y \implies 2x = 2 \frac{dy}{dx} \implies x = m \implies (x, y) = (m, \frac{m^2}{2}) \]

\[ y^2 = -4x \implies 2y \frac{dy}{dx} = -4 \implies y = \frac{-2}{m} \implies (x, y) = (\frac{-1}{m^2}, \frac{-2}{m}) \]

\[ y = mx + b \]
\[ x^2 = 2y \quad \implies 2x = 2 \frac{dy}{dx} \quad \implies x = m \quad \implies (x, y) = \left( m, \frac{m^2}{2} \right) \]

\[ y^2 = -4x \quad \implies 2y \frac{dy}{dx} = -4 \quad \implies y = \frac{-2}{m} \quad \implies (x, y) = \left( \frac{-1}{m^2}, \frac{-2}{m} \right) \]

\[ y = mx + b \]

\[ \implies \frac{m^2}{2} = mm + b \]

\[ \implies \frac{-2}{m} = m \frac{-1}{m^2} + b \]
\[ x^2 = 2y \quad \Rightarrow \quad 2x = 2 \frac{dy}{dx} \quad \Rightarrow \quad x = m \quad \Rightarrow \quad (x, y) = (m, \frac{m^2}{2}) \]

\[ y^2 = -4x \quad \Rightarrow \quad 2y \frac{dy}{dx} = -4 \quad \Rightarrow \quad y = \frac{-2}{m} \quad \Rightarrow \quad (x, y) = (\frac{-1}{m^2}, \frac{-2}{m}) \]

\[ y = mx + b \]

\[ \Rightarrow \quad \frac{m^2}{2} = mm + b \]

\[ \Rightarrow \quad \frac{-2}{m} = m \frac{-1}{m^2} + b \]

\[ \Rightarrow \quad b = -\frac{m^2}{2} = -\frac{1}{m} \]
\[ x^2 = 2y \quad \implies \quad 2x = 2\frac{dy}{dx} \quad \implies \quad x = m \quad \implies \quad (x, y) = (m, \frac{m^2}{2}) \]

\[ y^2 = -4x \quad \implies \quad 2y\frac{dy}{dx} = -4 \quad \implies \quad y = \frac{-2}{m} \quad \implies \quad (x, y) = \left(\frac{-1}{m^2}, \frac{-2}{m}\right) \]

\[ y = mx + b \]

\[ \implies \frac{m^2}{2} = mm + b \]

\[ \implies \frac{-2}{m} = m \frac{-1}{m^2} + b \]

\[ \implies m^3 = 2 \]
\[ x^2 = 2y \quad \implies 2x = 2 \frac{dy}{dx} \quad \implies x = m \quad \implies (x, y) = \left( m, \frac{m^2}{2} \right) \]

\[ y^2 = -4x \quad \implies 2y \frac{dy}{dx} = -4 \quad \implies y = \frac{-2}{m} \quad \implies (x, y) = \left( \frac{-1}{m^2}, \frac{-2}{m} \right) \]

\[ y = mx + b \quad \implies \frac{m^2}{2} = mm + b \]
\[ \implies \frac{-2}{m} = m \frac{-1}{m^2} + b \]
\[ \implies b = -\frac{m^2}{2} = -\frac{1}{m} \]

\[ \implies m^3 = 2 \quad \implies m = \sqrt[3]{2} \]
References:

• Euclid’s elements: http://aleph0.clarku.edu/~djoyce/java/elements/elements.html (with explanations of the proofs and a geometry applet) and http://www.math.ubc.ca/~cass/Euclid/byrne.html (“in which coloured diagrams and symbols are used instead of letters for the greater ease of learners”)

• Wikipedia page about the origami axioms: http://en.wikipedia.org/wiki/Huzita%E2%80%93Hatori_axioms


• Robert Lang’s webpage has loads of materials (including many of the photos of fancy origami): http://www.langorigami.com/. In particular, this paper has a lot a information: http://www.langorigami.com/science/math/hja/origami_constructions.pdf

• A TED talk about origami math things, but not quite the same content as this talk: http://www.ted.com/talks/robert_lang_folds_way_new_origami.html

• Folding a regular heptagon!
  http://www.math.sjsu.edu/~alperin/TotallyRealHeptagon.pdf