1 Questions

Question 1.1. Compare \((R_1 \circ R_2) \circ X_A\) and \(R_1 \circ (R_2 \circ X_A)\).

Question 1.2. Which ones of the following operations are associative.

(i) Addition on the real numbers. Multiplication on the positive real numbers.

(ii) Division on the non-zero real numbers.

Question 1.3. Suppose an operation \(*\) on \(X\) has two identity elements \(e\) and \(f\). Use the identity property for both \(e\) and \(f\) to simplify \(e \ast f\) into two different expressions, and conclude that \(e = f\) necessarily.

Question 1.4. Let \(X, *\) and \(e\) as in the previous definition but assume that \(*\) is associative. Now let \(x \in X\) with two inverses \(y\) and \(z\). In other words, \(x \ast y = y \ast x = e\) and \(x \ast z = z \ast x = e\). Show that \(z = z \ast (x \ast y) = (z \ast x) \ast y\) and conclude that \(z = y\) and hence that an inverse under an associative operation is unique if it exists.

Question 1.5. Let \(X, *\) and \(e\) as in the previous definition, but this time assume that \(X\) is finite, i.e. it only has finitely many elements, and that \(*\) is associative. Now let \(x \in X\) and \(y \in X\) such that \(x \ast y = e\). Consider the map \(f : X \rightarrow X\) defined by \(f(a) = y \ast a\) for \(a \in X\).

(i) Show that \(f\) is one-to-one, a.k.a injective, by using an associativity trick on \(x \ast f(a)\)

(ii) Deduce that \(f\) is also onto because \(X\) is finite.

(iii) Deduce now that the previous statement implies that there exists an element \(z \in X\) with \(y \ast z = e\).

(iv) Show that \(x = x \ast (y \ast z)\) implies that \(x = z\) and \(y \ast x = e\).

Question 1.6. Does the set of all symmetries of a square form a group?

Question 1.7. Does the set of all permutation of \(n\) elements form a group?

Question 1.8. Which one of these sets with operations are Groups?

1. The natural numbers with addition? With multiplication? How about substraction?

2. Same question with the rational numbers?

Forming a new group from old groups?

Question 1.9. Let \((\mathbb{Z}, +)\) be the additive integer group. Is \(\mathbb{Z} \times \mathbb{Z} = \{(a, b) \mid a, b \in \mathbb{Z}\}\) a group? What would be its binary operation?

Question 1.10. Can you generalize this to any group \((G, \ast)\)?
**Question 1.11.** Let $M$ be the set of expressions of the form \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) where $a, b, c$ and $d$ are real numbers. Define the binary operation $+$ on $M$ by the following expression:

\[
\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{pmatrix}.
\]

Show that this operation makes $M$ into a group. What is the identity in this case?

**Question 1.12.** Let $O$ be the subset of $M$ with elements \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) such that $ad - bc \neq 0$, with operation $\ast$ defined:

\[
\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \ast \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + d_1c_2 & c_1b_2 + d_1d_2 \end{pmatrix}.
\]

Show that this makes $O$ into a group with identity. What is the identity in this case?

**Question 1.13.** What are the rotational symmetries of a 3-dimensional cube?