The Role of Mathematics in Understanding the Earth’s Climate

Andrew Roberts
Outline

• What is climate (change)?
• History of mathematics in climate science
• How do we study the climate?
• Dynamical systems
• Large-scale (Atlantic) ocean circulation
• Ice ages and the mid-Pleistocene transition
• Winter is coming?
Weather vs. Climate

- Conditions of the atmosphere over a short period of time (minutes - months)
  - Temp, humidity, precip, cloud coverage (today)
  - Snowfall on November 14, 2014
  - Heat wave in 2010
  - Hurricane

- How the atmosphere “behaves” over a long period of time
  - Average of weather over time and space (usually 30-yr avg)
  - Historical average November precipitation
  - Record high temperature
  - Average number and strength of tropical cyclones, annually
Weather vs. Climate

• Climate is what you expect, weather is what you get

• Can view climate as a probability distribution of possible weather
What is Climate Change?
What is Climate Change?
What is Climate Change?
What is Climate Change?
Precipitation

Dry areas get dryer, wet areas get wetter.

Climate scientists predict more floods and more droughts!
Mathematics and Climate Change

\[ \Delta T = \mathcal{Q}(1 - \alpha(T)) - \sigma T^4 \]

- Energy balance equation
- \( \mathcal{Q} \): incoming solar radiation
- \( (1 - \alpha(T)) \): proportion absorbed by the Earth
- \( \sigma T^4 \): heat re-radiated back to space
Energy Balance

\[ \Delta T = 0 \Rightarrow Q(1 - \alpha(T)) = \sigma T^4 \]
Greenhouse Effect

- Joseph Fourier attempted to calculate the average temperature of the Earth (c. 1820)

- Hypothesized what has come to be known as the “greenhouse effect” — something is trapping heat in the Earth’s atmosphere

- 50 years before Stefan-Boltzmann energy balance equation

- 75 years before Arrhenius quantified how much colder the Earth “should” be
Greenhouse Effect

\[ \Delta T = Q(1 - \alpha(T)) - \varepsilon \sigma T^4 \]
Energy Balance Cartoon

- Reflected Solar Radiation: 107 W/m²
- Incoming Solar Radiation: 342 W/m²
- Outgoing Longwave Radiation: 235 W/m²
- Emitted by Atmosphere: 165 W/m²
- Greenhouse Gases
- Absorbed by Surface: 168 W/m²
- Evapotranspiration: 78 W/m²
- Latent Heat: 78 W/m²
- Absorbed by Surface: 324 W/m²
- Thermals: 24 W/m²
- Surface Radiation: 390 W/m²
- Back Radiation: 324 W/m²
- Reflected by Clouds, Aerosol, and Atmospheric Gases: 77 W/m²
- Reflected by Surface: 30 W/m²
Ice Ages

- Mid-1700s: speculation that ice ages exist
- 1830s: A few geologists claim ice-ages happened, ideas rejected
- 1842: Joseph Adhémar (mathematician) is first to propose ice-ages caused by variation in solar radiation
Ice Ages

• 1870s: Geologists reach consensus that ice-ages occurred (James Croll)

• 1912-1924: Milutin Milankovic
  • Eccentricity (100 kyr) — Kepler 1609
  • Obliquity/Axial tilt (41 kyr) — Milankovic 1912
  • Precession (23 kyr) — Hipparchus 130 B.C.
Milankovic Cycles

- Eccentricity
- Obliquity
- Precession
Snowball Earth

\[
\frac{\partial T(y)}{\partial t} = Q_s(y)(1 - \alpha(y, \eta)) - (A + BT(y)) - C(T(y) - \bar{T}(\eta))
\]

- Budyko and Sellers (1969) describe spatially dependent energy balance model
- Assume Northern and Southern hemisphere symmetric
- Assume temperature is the same for fixed latitude \((y)\)
- Includes energy transport term
Snowball Earth

• 2 stable states of ice coverage:
  • Warm climate (like now - and even ice ages)
  • Snowball climate (entire Earth covered in ice)

• Dismissed as “mathematical artifact” until 1990s

• New consensus: 3 snowball events (all over 600 myr ago)
How do we study the climate?
How do we study the climate?
How do we study the climate?
How do we study the climate?

Models!
Model Hierarchies
Conceptual Models

• Examples: Energy balance models

• Typically model 1 or 2 processes/phenomena

• Large-scale average behavior

• Help explain climate to non-experts

• Motivate large experiments

• Pros:
  • Simple enough to be analyzed by a person
  • Can explore all possibilities
  • Intuition

• Cons:
  • Too simple to prove scientific results definitively
  • Adding more processes could destroy phenomenon
Intermediate Complexity and Process Models

- Some spatial resolution
- More processes (but not too many)
- Simple enough for some interpretation
- Too complex to analyze “by hand”
GCMs and ESMs

- Too complicated to interpret causality
- Too complicated to explore all possibilities (where do we look?)
- Millions of lines of code (bugs?)
- Expensive (financially and computationally)
- Treated as “experimental Earths”
- Useful for prediction*
Weather Prediction

Observation of Current State

Model
Weather Prediction

Observation of Current State

Model

Prediction (1 hour)
Weather prediction

Observation of Current State

Model → Prediction (1 hour) → Model → Prediction (2 hour)
Observations have error

Observed state

Actual (initial) state
Error grows

Observed state → Model → State after 1 hr

Actual (initial) state

1 hour prediction
and grows...

Observed state

Actual (initial) state

Model

State after 1 hr

1 hour prediction

Model

State after 2 hr

2 hour prediction
Lorenz Butterfly
Climate Prediction
Climate Prediction
Climate Prediction
Climate Prediction

Where do observations come in?
Confronting Models with Data
Confronting Models with Data
Confronting Models with Data
Data Assimilation
Data Assimilation
The Role of Mathematics in Climate Science

Field
- Observation Data

Lab
- Simulations as Experiments
- ESMs
- GCMs

Theory
- Conceptual Models

Math
Dynamical Systems

Derivative (from Calculus)
\[ \frac{dx}{dt} = f(t) \]

Example
\[ \frac{dx}{dt} = t^3 - t + k \]
\[ x(t) = \frac{t^4}{4} - \frac{t^2}{2} + kt + C \]
Dynamical Systems

Derivative (from Calculus)

\[ \frac{dx}{dt} = f(t) \]

Example

\[ \frac{dx}{dt} = t^3 - t + k \]

What if \[ \frac{dx}{dt} = f(x) \]?
More than one variable?

System of Differential Equations

\[ \dot{x} = f(x, y) \]
\[ \dot{y} = g(x, y) \]

Defines a Vector Field
Vector Fields

System of Differential Equations

\[ \dot{x} = y - x^3 + x \]
\[ \dot{y} = x - 2y + k \]

Defines a Vector Field
Equilibrium Points

\[ \dot{x} = y - x^3 + x \]
\[ \dot{y} = x - 2y + k \]

Equilibrium points occur when

\[ \dot{x} = 0 \]
\[ \dot{y} = 0 \]
\[ \dot{x} = y - x^3 + x \]
\[ \dot{y} = x - 2y + k \]

Even if equations can’t be solved, we can understand

Qualitative Behavior
Varying $k$

\[
\begin{align*}
\dot{x} &= y - x^3 + x \\
\dot{y} &= x - 2y + 0
\end{align*}
\]

\[
\begin{align*}
\dot{x} &= y - x^3 + x \\
\dot{y} &= x - 2y - 2
\end{align*}
\]
Bifurcation in Algebra I

Quadratic equation

$$ax^2 + bx + c = 0$$
Bifurcation in Algebra I

Quadratic equation

\[ ax^2 + bx + c = 0 \]

Bifurcation parameter:
Discriminant

\[ b^2 - 4ac > 0 \]

2 Real Roots
Bifurcation in Algebra I

Quadratic equation

\[ ax^2 + bx + c = 0 \]

Bifurcation parameter: Discriminant

\[ b^2 - 4ac > 0 \]

2 Real Roots

No qualitative change for small change in equation
Bifurcation in Algebra I

Quadratic equation

\[ ax^2 + bx + c = 0 \]

Bifurcation parameter: Discriminant

\[ b^2 - 4ac < 0 \]

0 Real Roots

Big enough change in system leads to qualitatively different solutions
Bifurcation in Algebra I

Quadratic equation

\[ ax^2 + bx + c = 0 \]

Bifurcation parameter: Discriminant

\[ b^2 - 4ac = 0 \]

1 Real Root

Bifurcation occurs when solutions collide
Bifurcations as Tipping Points
Bifurcations as Tipping Points
Bifurcations as Tipping Points
Bifurcations as Tipping Points
Hysteresis
Bifurcation vs. Intrinsic Dynamics

- Idea of bifurcations assumes modeler has control over how parameters change — i.e., do NOT depend on state of system

- Snowball Earth: bifurcation “parameter” depends on GHGs (which in turn depend on temperature and ice)

- How does behavior change?
Fast/Slow Dynamics

- Fast variable is like state of system as before
- Slow variable is acts partly like parameter, partly like state variable
- Example of parameter: Milankovic cycles depend only on time (influence climate, but not influenced by climate)
- Examples of slow variable: GHGs, Ice coverage

\[
\begin{align*}
\dot{x} &= f(x; \lambda) \\
\lambda(t) &= \tilde{g}(t) \\
\dot{x} &= f(x, y) \\
\dot{y} &= \varepsilon g(x, y) \\
\varepsilon &\ll 1
\end{align*}
\]
Picturing the difference

\[ \dot{x} = f(x; \lambda) \]
\[ \lambda(t) = \tilde{g}(t) \]

\[ \dot{x} = f(x, y) \]
\[ \dot{y} = \varepsilon g(x, y) \]
\[ \varepsilon \ll 1 \]
Example in Ocean Circulation
Stommel’s Circulation Model

Figure: Schematic of Stommel’s model (1961)—from Saha (2011).

Circulation variable: $\psi$
Stommel’s Circulation Model

Model Reduces:

\[ x \sim T_e - T_p \rightarrow 1 \]

Get one state variable:

\[ y \sim S_e - S_p \]

Bifurcation parameter:

\[ \mu \sim \frac{\Delta S^A}{\Delta T^A} \]

\[ \dot{y} = \mu - y - A|1 - y|y \]
\( \mu \rightarrow \text{slow variable} \)

\[ \dot{y} = \mu - y - A|1 - y|y \]

\[ \dot{\mu} = \delta_0 (\lambda - y) \]

(a) Stable periodic orbit when \( A = 5, \lambda = 0.8, \text{and} \ \delta = 0.1 \)

(b) Time series for \( \psi \) for the trajectory in

(c) Canard trajectory when \( A = 1.1, \lambda = 0.995, \text{and} \ \delta_0 = 0.01 \).

(d) Super-explosion when \( A = 1.5, \lambda = 0.995, \text{and} \ \delta_0 = 0.01 \).
Mixed-mode Oscillations

- 2D dynamical systems can have up to 3 end states:
  - Fixed equilibrium
  - Periodic equilibrium (with fixed amplitude and period)
  - Run-away behavior
- MMOs have big and small oscillations—need 3D system!
  - 3D dynamics much more complicated (chaos)
Ice Ages over the last 400 kyr

\begin{align*}
\dot{x} &= y - x^3 + 3x - k \\
\dot{y} &= \varepsilon[p(x - a)^2 - b - my - (\lambda + y - z)] \\
\dot{z} &= \varepsilon r(\lambda + y - z)
\end{align*}
Ice Ages over the last 400 kyr

\[
\begin{align*}
\dot{x} &= y - x^3 + 3x - k \\
\dot{y} &= \varepsilon [p(x - a)^2 - b - my - (\lambda + y - z)] \\
\dot{z} &= \varepsilon r (\lambda + y - z)
\end{align*}
\]

\(x \sim \text{ice volume}\) \hspace{2cm} \(z \sim \text{oceanic carbon}\)

\(y \sim \text{atmospheric carbon}\)
Ice Ages over the last 400 kyr

\[
\begin{align*}
\dot{x} &= y - x^3 + 3x - k \\
\dot{y} &= \varepsilon[p(x - a)^2 - b - my - (\lambda + y - z)] \\
\dot{z} &= \varepsilon r(\lambda + y - z)
\end{align*}
\]

Fast  Slow
Ice Ages over the last 400 kyr

\[ \dot{x} = y - x^3 + 3x - k \]
\[ \dot{y} = \varepsilon[p(x - a)^2 - b - my - (\lambda + y - z)] \]
\[ \dot{z} = \varepsilon r(\lambda + y - z) \]

Change in ice volume depends on temperature, but temperature depends on the amount of ice and how much GHGs are in the atmosphere.
Ice Ages over the last 400 kyr

\[ \dot{x} = y - x^3 + 3x - k \]
\[ \dot{y} = \varepsilon[p(x - a)^2 - b - my - (\lambda + y - z)] \]
\[ \dot{z} = \varepsilon r(\lambda + y - z) \]

Land-atmosphere carbon flux
Ice Ages over the last 400 kyr

\[ \dot{x} = y - x^3 + 3x - k \]
\[ \dot{y} = \varepsilon[p(x - a)^2 - b - my - (\lambda + y - z)] \]
\[ \dot{z} = \varepsilon r(\lambda + y - z) \]

Ocean-atmosphere atmosphere carbon flux
Ice Ages over the last 400 kyr
Ice Ages over the last 400 kyr
Ice Ages over the last 400 kyr
Ice Ages over the last 400 kyr
Ice Ages over the last 400 kyr
El Niño-Southern Oscillation
How does ENSO work?
The Data
Predicting ENSO

August predictions

Talk of an El Niño year cools, but don’t despair yet about winter

While a 'super' El Niño looks to be off the table, what does develop this year might not deliver what many Canadians are hoping for.

Don’t dismiss a 2014 ‘super’ El Niño just yet
Predicting ENSO

- August Prediction
  - ？
- September prediction
  - Probability of ENSO: low
- October Prediction
  - Probability of ENSO: 0.68
  - November Prediction
    - 58% chance of ENSO
    - Normal to weak ENSO
ENSO

\[
\begin{align*}
\dot{x} &= \epsilon(x^2 - ax) + x \left[ x + y - nz + d - c \left( x - \frac{x^3}{3} \right) \right] \\
\dot{y} &= -\epsilon(ay + x^2) \\
\dot{z} &= m \left( k - z - \frac{x}{2} \right)
\end{align*}
\]
ENSO

\[
\dot{x} = \varepsilon(x^2 - ax) + x \left[ x + y - nz + d - c \left( x - \frac{x^3}{3} \right) \right]
\]

\[
\dot{y} = -\varepsilon(ay + x^2)
\]

\[
\dot{z} = m \left( k - z - \frac{x}{2} \right)
\]

- $x$: temperature gradient
- $y$: temp of W Pacific
- $z$: thermocline dept in W Pacific
ENSO

\[
\begin{align*}
\dot{x} &= \varepsilon(x^2 - ax) + x \left[ x + y - nz + d - c \left( x - \frac{x^3}{3} \right) \right] \\
\dot{y} &= -\varepsilon(ay + x^2) \\
\dot{z} &= m \left( k - z - \frac{x}{2} \right)
\end{align*}
\]

Upwelling feedback
\[ \dot{x} = \varepsilon (x^2 - ax) + x \left[ x + y - nz + d - c \left( x - \frac{x^3}{3} \right) \right] \]
\[ \dot{y} = -\varepsilon (ay + x^2) \]
\[ \dot{z} = m \left( k - z - \frac{x}{2} \right) \]

Thermocline adjustment
ENSO

\[ \dot{x} = \varepsilon(x^2 - ax) + x \left[ x + y - nz + d - c \left( x - \frac{x^3}{3} \right) \right] \]

\[ \dot{y} = -\varepsilon(ay + x^2) \]

\[ \dot{z} = m \left( k - z - \frac{x}{2} \right) \]

Advection
ENSO
Simulation 1
Simulation 1
Simulation 2
Simulation 2
Model Output

Cubic Approximation - Dimensionalized

- Red: T1
- Blue: T2

Graph showing variations in T1 and T2 with 'h' indicating another parameter.
Winter?