# Links We Almost Missed Between Delannoy Numbers and Legendre Polynomials

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# Delannoy numbers

$$d_{i,j} = d_{i-1,j} + d_{i,j-1} + d_{i-1,j-1}$$

$d_{i,j} :=$

| $i \backslash j$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 3 | 5 | 7 | 9 |
| 2 | 1 | 5 | 13 | 25 | 41 |
| 3 | 1 | 7 | 25 | 63 | 129 |
| 4 | 1 | 9 | 41 | 129 | 321 |

They count the number of lattice paths from $(0,0)$ to $(m,n)$ using only steps $(1,0)$, $(0,1)$, and $(1,1)$.

$\Rightarrow d_{n,n} = \sum_{j=0}^{n} \binom{n}{j}\binom{n+j}{j}$.

(Defined by Henri Delannoy (1895), Sulanke has $\geq 29$ interpretations.)

## A mysterious relation with the Legendre polynomials

Good (1958), Lawden (1952), Moser and Zayachkowski (1963) observed that

$$d_{n,n} = P_n(3),$$

where $P_n(x)$ is the $n$-th Legendre polynomial.

There has been a consensus that this link is not very relevant.

Banderier and Schwer (2004): "there is no "natural" correspondence between Legendre polynomials and these lattice paths."

Sulanke (2003): "the definition of Legendre polynomials does not appear to foster any combinatorial interpretation leading to enumeration".

("Your nose is big." (Rostand: Cyrano de Bergerac))

## Jacobi and Legendre polynomials

Usual definition of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$:

$$\begin{aligned} P_n^{(\alpha,\beta)}(x) &= (-2)^{-n}(n!)^{-1}(1-x)^{-\alpha}(1+x)^{-\beta} \\ & \frac{d^n}{dx^n}\left((1-x)^{n+\alpha}(1+x)^{n+\beta}\right). \end{aligned}$$

$\alpha, \beta > -1$ "for integrability purposes", $\alpha = \beta = 0$ gives Legendre.

The formula below extends to all $\alpha, \beta \in \mathbb{C}$ (see Szegő (4.21.2)):

$$P_n^{(\alpha,\beta)}(x) = \sum_j \binom{n+\alpha+\beta+j}{j}\binom{n+\alpha}{n-j}\left(\frac{x-1}{2}\right)^j .$$

Substitute $\alpha = \beta = 0$:

$$P_n^{(0,0)}(x) = \sum_j \binom{n+j}{j}\binom{n}{j}\left(\frac{x-1}{2}\right)^j$$

is the $n$-th Legendre polynomial.

# Part I: Balanced simplicial complexes

## Asymmetric Delannoy numbers

$\widetilde{d}_{m,n} :=$

| $m \backslash n$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 1 | 2 | 4 | 8 | 16 |
| 1 | 1 | 3 | 8 | 20 | 48 |
| 2 | 1 | 4 | 13 | 38 | 104 |
| 3 | 1 | 5 | 19 | 63 | 192 |
| 4 | 1 | 6 | 26 | 96 | 321 |

$\widetilde{d}_{m,n}$ is the number of lattice paths from $(0, 0)$ to $(m, n+1)$ having steps $(x, y) \in \mathbb{N} \times \mathbb{P}$.

(Variant of A049600 in the On-Line Encyclopedia of Integer Sequences.)

**Lemma 1** *The asymmetric Delannoy numbers satisfy*

$$\widetilde{d}_{m,n} = \sum_{j=0}^{n} \binom{n}{j} \binom{m+j}{j}.$$

**Proof:** We are enumerating sequences $(0,0) = (x_0, y_0), (x_1, y_1), \ldots, (x_j, y_j), (x_{j+1}, y_{j+1}) = (m, n+1)$, where $0 \le j \le n$, $0 = x_0 \le x_1 \le \cdots \le x_j \le x_{j+1} = m$, and $0 = y_0 < y_1 < \cdots < y_j < y_{j+1} = n+1$. For a given $j$ there are $\binom{m+j}{j}$ ways to choose $0 = x_0 \le x_1 \le \cdots \le x_j \le x_{j+1} = m$ and $\binom{n}{j}$ ways to choose $0 = y_0 < y_1 < \cdots < y_j < y_{j+1} = n+1$. $\diamond$

Since

$$P_n^{(0,\beta)}(x) = \sum_j \binom{n+\beta+j}{j} \binom{n}{j} \left(\frac{x-1}{2}\right)^j,$$

we get

$$\widetilde{d}_{m,n} = P_n^{(0,m-n)}(3) \quad \text{for } m \ge n$$

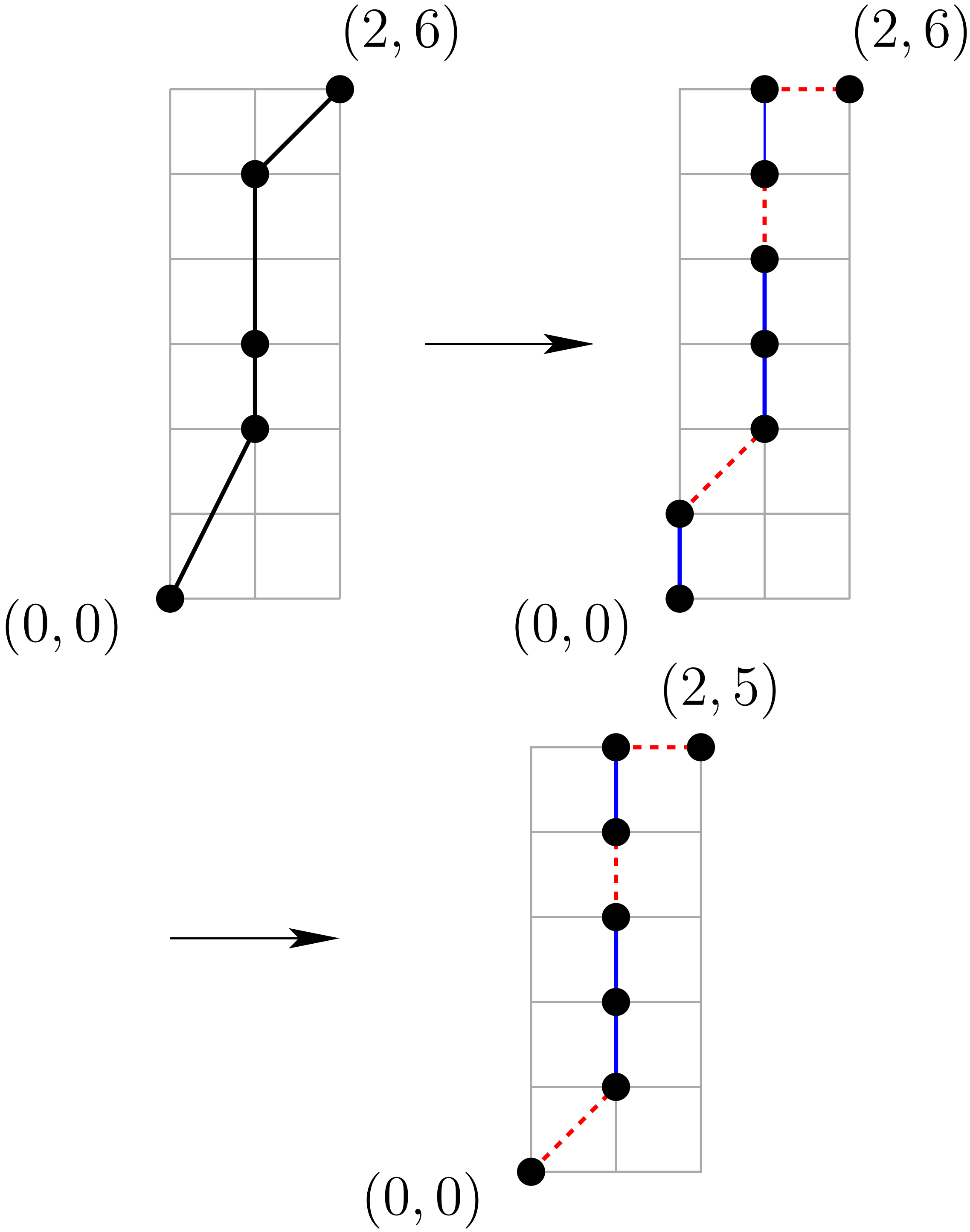
because $\dfrac{3-1}{2} = 1$.

## A related 2-colored path enumeration problem

**Proposition 1** *For any* $(m,n) \in \mathbb{N} \times \mathbb{N}$ *the number* $\widetilde{d}_{m,n}$ *also enumerates all* 2*-colored lattice paths from* $(0,0)$ *to* $(m,n)$ *satisfying the following:*

(i) *Each step is either a blue* $(0,1)$ *or a red* $(x,y) \in \mathbb{N} \times \mathbb{N} \setminus \{(0,0)\}$.

(ii) *At least one of any two consecutive steps is a blue* $(0,1)$.




($m = 2$ and $n = 5$.)

## Simplicial complexes

*simplicial complex* $\triangle$*:* family of subsets of $V$, such that $\{v\} \in \triangle$ for all $v \in V$ and every subset of a $\sigma \in \triangle$ belongs to $\triangle$.

*face:* $\sigma \in \triangle$, *dimension:* $\dim(\sigma) = |\sigma| - 1$.

$f$*-vector:* $(f_{-1}, \ldots, f_{n-1})$, where $f_{j-1}$ is the number of $(j-1)$-dim faces.

$h$*-vector:* $(h_0, \ldots, h_n)$ given by

$$h_i = \sum_{j=0}^{i} (-1)^{i-j} \binom{n-j}{i-j} f_{j-1}.$$

*balanced complex:* $(n-1)$-dimensional, vertices may be colored using $n$-colors.

*flag* $f$*-vector*: $(f_S : S \subseteq \{1, 2, \ldots, n\})$

*flag* $h$*-vector*: $(h_S : S \subseteq \{1, 2, \ldots, n\})$, given by

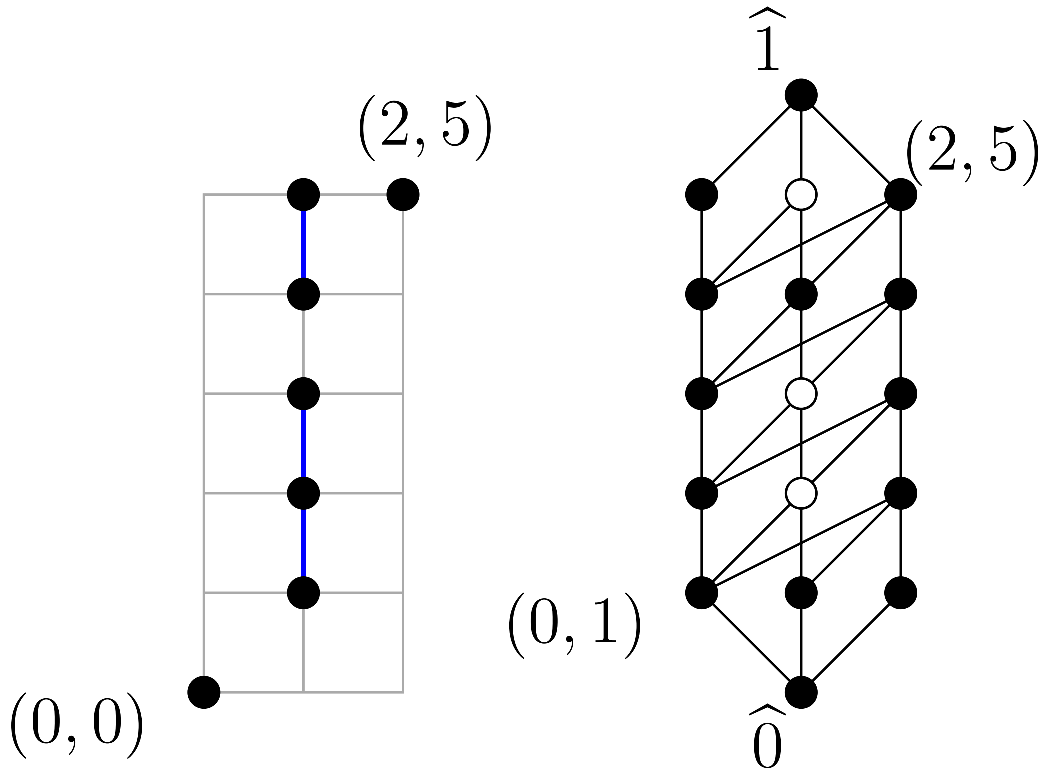
$$h_S = \sum_{T \subseteq S} (-1)^{|S \setminus T|} f_T.$$

**Theorem (Björner-Frankl-Stanley):**
$(\beta_S : S \subseteq \{1, 2, \ldots, n\})$ is the flag $h$-vector of some $(n-1)$-dimensional balanced Cohen-Macaulay complex $\Leftrightarrow$ it is the flag $f$-vector a colored simplicial complex.

## The order complex of a Jacobi poset

$(i)$ For each $q \in \{1, \ldots, n\}$, $P_n^\beta$ has $n+\beta+1$ elements of rank $q$, they are labeled $(0,q)$, $(1,q)$, $\ldots$, $(n+\beta,q)$.

$(ii)$ Given $(p,q)$ and $(p',q')$ in $P_n^\beta \setminus \{\widehat{0},\widehat{1}\}$ we set $(p,q) < (p',q')$ iff. $p \le p'$ and $q < q'$.




The Jacobi poset $P_5^{-3}$

## Why the name Jacobi?

$$f_{j-1}\left(\triangle\left(P_n^{\beta}\setminus\{\widehat{0},\widehat{1}\}\right)\right)=\binom{n}{j}\binom{n+\beta+j}{j}.$$

$$\sum_{j=0}^{n} f_{j-1}\left(\triangle\left(P_n^{\beta}\setminus\{\widehat{0},\widehat{1}\}\right)\right)\cdot\left(\frac{x-1}{2}\right)^{j}=P^{(0,\beta)}(x)$$

for $\beta\geq 0$.

## Connection with the asymmetric Delannoy numbers

*balanced join operation:* Let $\triangle_1$ and $\triangle_2$ be pure, balanced, of the same dimension. Assume the balanced colorings $\lambda_1$ and $\lambda_2$ use the same set of colors. Then, for $\lambda=(\lambda_1,\lambda_2)$,

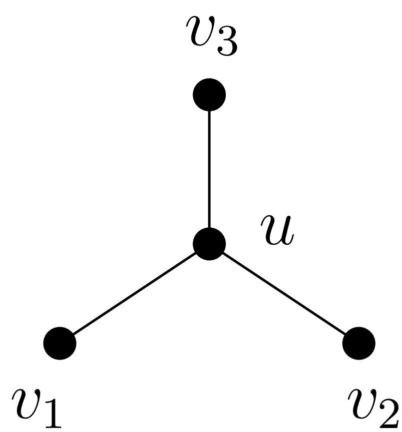
$$\triangle_1 *_\lambda \triangle_2=\{\sigma\cup\tau:\sigma\in\triangle_1,\tau\in\triangle_2,\lambda_1(\sigma)\cap\lambda_2(\tau)=\emptyset\}$$

is the *balanced join* of $\triangle_1$ and $\triangle_2$ with respect to $\lambda$.

**Theorem 1** *$\widetilde{d}_{m,n}$ is the number of facets in the balanced join $\triangle\left(P_n^{m-n}\setminus\{\widehat{0},\widehat{1}\}\right) *_\lambda \triangle^{n-1}$.*

## Properties of the balanced join operation

*It depends on the coloring chosen:*




$\triangle *_\lambda \triangle$ has $1 \cdot 3 + 3 \cdot 1 = 6$ edges if $\lambda_1 = \lambda_2$, but $1 \cdot 1 + 3 \cdot 3 = 10$ edges if $\lambda_1 \neq \lambda_2$.

**Theorem 2** *The flag h-vector of the balanced join $\triangle *_\lambda \triangle^{n-1}$ of a balanced $(n-1)$-dimensional simplicial complex with an $(n-1)$-simplex $\triangle^{n-1}$ satisfies*

$$h_S\left(\triangle *_\lambda \triangle^{n-1}\right) = f_S\left(\triangle\right).$$

## Balanced join and the C-M property

**Theorem 3** *If $\triangle_1$ and $\triangle_2$ are balanced Cohen-Macaulay and their balanced join exists, then $\triangle_1 *_\lambda \triangle_2$ is also Cohen-Macaulay.*

The proof uses Reisner's Criterion:

**Theorem 4 (Reisner)** *$\triangle$ is Cohen-Macaulay if and only if for all $\sigma \in \triangle$ and $i < \dim \mathrm{lk}_\triangle(\sigma)$ we have $\widetilde{H}_i(\mathrm{lk}_\triangle(\sigma), k) = 0$. Here $\widetilde{H}_i$ denotes the $i$-th reduced homology group of the appropriate oriented chain complex.*

*link:* $\mathrm{lk}_\triangle(\tau) := \{\sigma \in \triangle : \sigma \cap \tau = \emptyset, \sigma \cup \tau \in \triangle\}$.
*oriented $j$-simplex:* $[v_0, \ldots, v_j]$, $\{v_0, \ldots, v_j\} \in \triangle$, permuting elements induces multiplying with the sign of the permutation. These generate $C_j(\triangle)$.
*boundary map:* $\partial_j : C_j(\triangle) \to C_{j-1}(\triangle)$, given by

$$\partial_j [v_0, \ldots, v_j] = \sum_{i=0}^{j} (-1)^i [v_0, \ldots, \widehat{v}_i, \ldots, v_j].$$

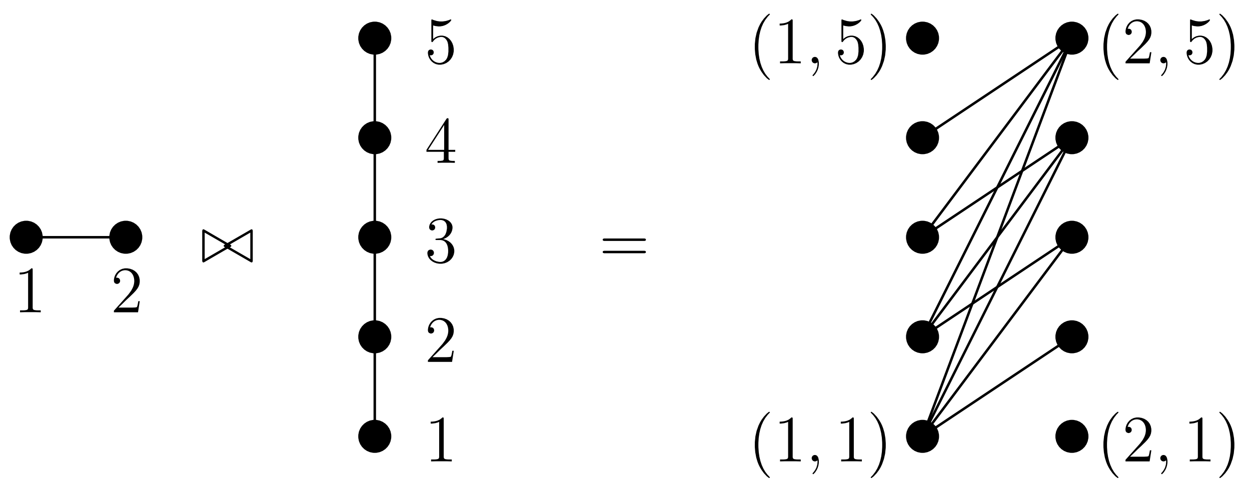
## More on Jacobi posets

**Proposition 2** *The order complex* $\triangle(P_n^\beta \setminus \{\widehat{0}, \widehat{1}\})$ *associated to the Jacobi poset* $P_n^\beta$ *is Cohen-Macaulay.*

Proof uses *EL-labelings.*

**Proposition 3** *The flag h-vector of* $\triangle(P_n^\beta \setminus \{\widehat{0}, \widehat{1}\})$*, colored by the rank function, equals the flag f-vector of* $\triangle(C_{n+\beta-1} \bowtie C_{n-1})$*, with respect to the coloring induced by the rank function of the second coordinate.*

*strict direct product* $P \bowtie Q$*:* the set $P \times Q$, ordered by $(p, q) < (p', q')$ if $p < p'$ and $q < q'$.



*right-strict direct product* $P \rtimes Q$ the set $P \times Q$, ordered by $(p, q) < (p'q')$ if $p \leq p'$ and $q < q'$.

**Proposition 4** $P_n^\beta \setminus \{\widehat{0}, \widehat{1}\}$ *is isomorphic to* $C_{n+\beta} \rtimes C_{n-1}$.

**Proposition 5** *Assume* $P$ *is an arbitrary poset and* $Q$ *is a graded poset of rank* $n+1$*. Then* $P \rtimes (Q \setminus \{\widehat{0}, \widehat{1}\})$ *may be turned into a graded poset of rank* $n+1$ *by adding a unique minimum element* $\widehat{0}$ *and a unique maximum element* $\widehat{1}$*. The rank function may be taken to be the rank function of* $Q$ *applied to the second coordinate.*

**Conjecture 1** *If* $P$ *is a poset with a Cohen-Macaulay order complex and* $Q$ *is a graded Cohen-Macaulay poset then* $P \rtimes (Q \setminus \{\widehat{0}, \widehat{1}\}) \cup \{\widehat{0}, \widehat{1}\}$ *is a graded Cohen-Macaulay poset.*

## The half-strict fact

**Theorem 5** *Assume that $P$ is any poset whose order complex has a non-negative $h$-vector and that $Q$ is a graded posets with a non-negative flag $h$-vector. Then the flag $h$-vector of $P \rtimes (Q \setminus \{\widehat{0}, \widehat{1}\}) \cup \{\widehat{0}, \widehat{1}\}$ is non-negative.*

In fact:

$h_S(P \rtimes (Q \setminus \{\widehat{0}, \widehat{1}\}) \cup \{\widehat{0}, \widehat{1}\}) =$

$$\sum_{T \subseteq S} h_T(Q) \sum_{i=0}^{\min(d,|R|)} h_i(P) \binom{d + |T| - i - 1}{|S| - i}.$$

# Part II: The “Legendrotope” of dimension $n$

**Central sections of centrally symmetric polytopes**

$P \subset \mathbb{R}^n$ is a centrally symmetric polytope, $H$ is given by $\sum_{i=1}^n \lambda_i x_i = \langle \lambda \mid x \rangle = 0$. $Q := P \cap H$ is a *non-degenerate central section* if $Q \cap V(P) = \emptyset$.

Then $V(P) = V_+(P) \uplus V_-(P)$ where
$V_+(P) := \{(x_1, \ldots, x_n) \in V(P) \, : \, \langle \lambda \mid x \rangle > 0\}$ and
$V_-(P) := \{(x_1, \ldots, x_n) \in V(P) \, : \, \langle \lambda \mid x \rangle < 0\}$. Each vertex of $Q$ is of the form $H \cap [u, -v]$.

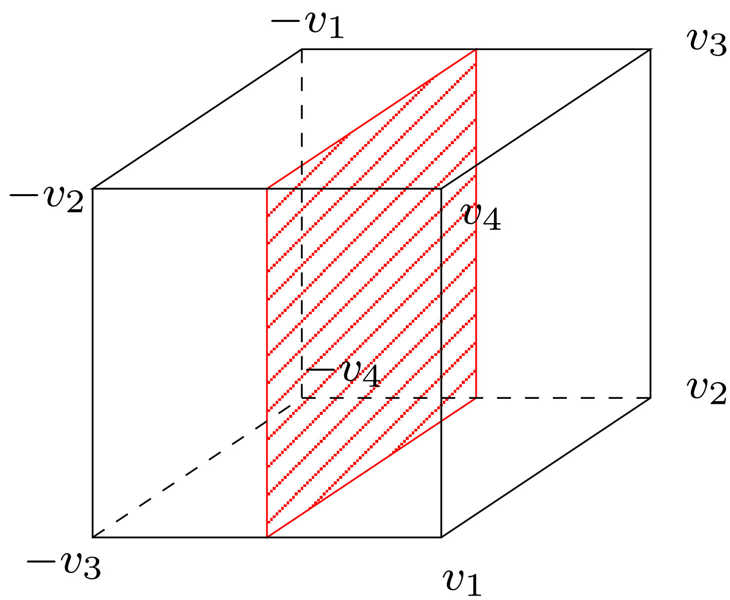
Define the graph $G = G(P, H)$ on $V(G) := V_+(P)$ by letting $(u, v)$ be a directed edge in $G$ exactly when $[u, -v] \cap H$ is a vertex of $Q$. This graph contains no loops.

**Definition 1** *Let $G$ be a directed graph with no multiple edges. Let $S$ and $T$ be disjoint subsets of $V(G)$. The* directed restriction of $G$ to $(S, T)$ *is the digraph with vertex set $S \cup T$ with edge set* $\{(s, t) \in E(G) \, : \, s \in S, t \in T\}$.

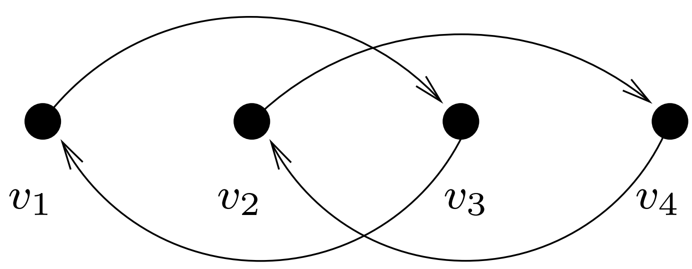
**Proposition 6** *Assume the vertices of $Q = P \cap H$ are represented by the edges of the graph $G = G(P, H)$. Given a face $F$ of $P$, the vertices contained in the face $F \cap H$ of $Q$ are represented by the edges in the directed restriction of $G(P, H)$ to $(V_+(F), -V_-(F))$.*

## An Example

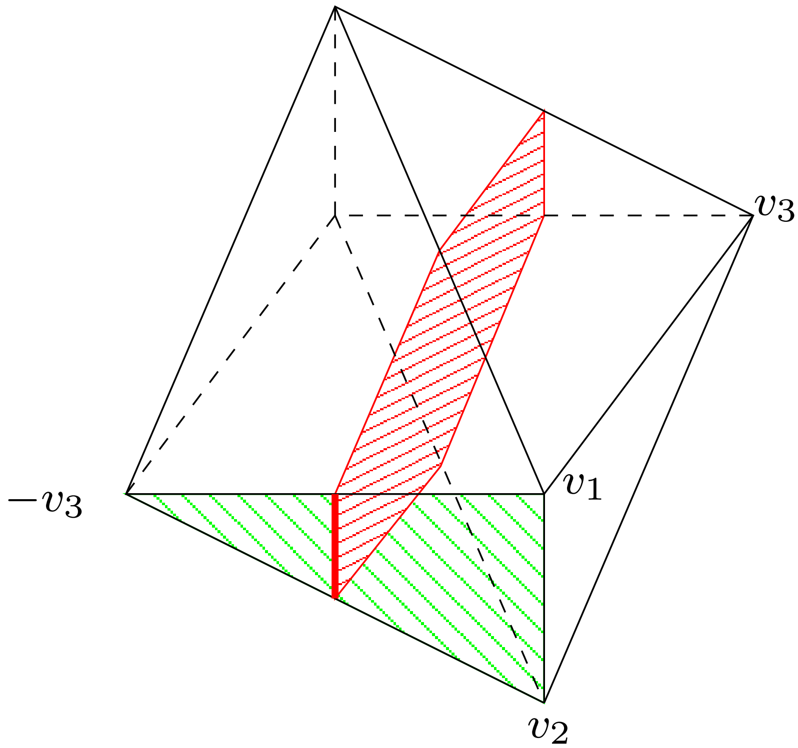
$P$ is the 3-dimensional cube, $H$ contains the midpoints of some parallel edges:

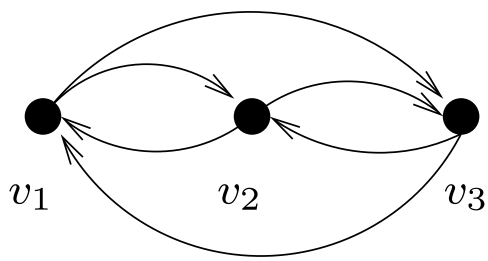



Here is the associated graph:

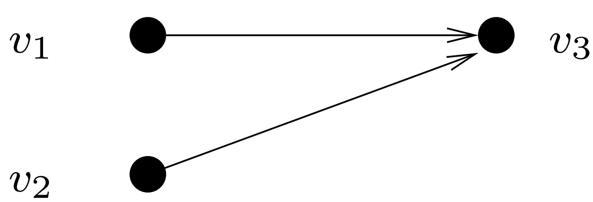


# The case of a simplicial $P$




The directed restriction of $G(P, H)$ to $(V_+(F), -V_-(F))$ is a complete bipartite graph, with each edge directed towards its endpoint in $V_-(F)$.



## Face structure

The following Lemma generalizes a result of Kapranov, Postnikov, and Zelevinski. (Stated for undirected graphs, trees, convex hulls of midpoints of edges.)

**Lemma 2** *Let $F \subset P$ be a face of $P$. A subset $S$ of the edges of the directed restriction of $G(P,H)$ to $(V_+(F), -V_-(F))$ represents a simplex if and only if, disregarding the orientation of the edges, the set $S$ contains no circuit.*

## The Legendre polytope

**Definition 2** *We define the* Legendre polytope $\mathcal{L}_n$ *as the non-degenerate central section of $2\mathcal{O}_{n+1}$ with the hyperplane*

$$H_n := \left\{ (x_0, \ldots, x_n) \,:\, \sum_{i=0}^{n} x_i = 0 \right\}.$$

**Lemma 3** *A set $S \subset \{(e_i, e_j) \,:\, i \neq j\}$ of edges represents all vertices in a facet of the boundary $\partial\mathcal{L}_n$ of $\mathcal{L}_n$ if and only if there is a proper subset $A$ of $A \subset \{e_0, \ldots, e_n\}$ such that $S$ consists of all edges starting in $A$ and ending in $\{e_0, \ldots, e_n\} \setminus A$.*

## Connection to root polytopes

- The convex hull of $\mathbf{0}$ and $\{e_i - e_j \ : \ i < j\}$ is the *root polytope* $P_{A_n^+}$ (Gelfand, Graev and Postnikov [9])
- $P_{A_n^+}$ is also known as the "Catalanotope" (Stanley).
- More general root polytopes were studied by Postnikov [17].
- Extensions to the root systems $B_n$, $C_n$, and $D_n$ were studied by Wungkum Fong [8].

Since all vertices of $\mathcal{L}_n$ not belonging to $P_{A_n^+}$ form $V(-P_{A_n^+}) \setminus \{\mathbf{0}\}$, we may think of $\mathcal{L}_n$ as the convex hull of $P_{A_n^+}$ and $-P_{A_n^+}$. The set of facets of $\partial\mathcal{L}_n$ may be partitioned into three classes:

1. the facets of $P_{A_n^+}$ not containing the origin;
2. the facets of $-P_{A_n^+}$ not containing the origin;
3. facets which contain at least one vertex of $P_{A_n^+}$ and one vertex of $-P_{A_n^+}$.

## Pulling triangulations of $\partial\mathcal{L}_n$

**Fact:** Facets in any triangulation of $P_{A_n^+}$ are associated with admissible trees on the vertex set $\{e_0, \ldots, e_n\}$.

**Translation**: Facets correspond to maximal cycle free sets of edges in directed restrictions $(S, T)$ such that $S \uplus T = V$.

Two special triangulations introduced by Gelfand, Graev and Postnikov are induced by the *standard* and *antistandard* trees. They correspond to pulling triangulations with respect to the lex and the revlex order.

**Definition 3** *The* revlex order *on* $V(P_{A_n^+}) \setminus \{\mathbf{0}\}$ *is defined by setting* $(e_i, e_j) < (e_k, e_l)$ *if* $j < l$ *or* $j = l$ *and* $i > k$. *The* lexicographic order *on* $V(P_{A_n^+}) \setminus \{\mathbf{0}\}$ *is defined by setting* $(e_i, e_j) < (e_k, e_l)$ *if* $i < k$ *or* $i = k$ *and* $j < l$.

**Proposition 7 (Stanley)** *Suppose that one of the vertices of* $P$ *is the origin and that the matrix whose rows are the vertices of* $P$ *is totally unimodular. Let* $<$ *be any ordering on* $V(P)$ *such that the origin is the least vertex with respect to* $V(P)$. *Then* $<$ *is compressed.*

**Theorem 6 (Heller)** *The incidence matrix of a directed graph is totally unimodular.*

## Consequences

**Corollary 1** *Not only the (anti-)standard triangulation, but all pulling triangulations of* $P_{A_n^+}$ *starting with pulling at the origin, are compressed.*

**Corollary 2** *Not only the (anti-)standard triangulation, but all pulling triangulations of* $\partial\mathcal{L}_n$ *are compressed.*

**Theorem 7** *The number of* $(j-1)$*-dimensional faces in any pulling triangulation of* $\partial\mathcal{L}_n$ *that uses only the vertices of* $\mathcal{L}_n$ *is*

$$f_{j-1}\left(\triangle_{<}\left(\partial\mathcal{L}_n\right)\right) = \binom{n+j}{j}\binom{n}{j}.$$

This is easy to show for the lexicographic order.

$$F_{\triangle}(x) := \sum_{j=0}^{d} f_{j-1}\left(\frac{x-1}{2}\right)^{j}.$$

**Corollary 3** *The* $F$*-polynomial of any pulling triangulation of* $\partial\mathcal{L}_n$ *that uses only the vertices of* $\mathcal{L}_n$ *is* $P_n(x)$*, the* $n$*-th Legendre polynomial.*

## How about the "Catalanotope"?

**Theorem 8** *The number of $(j-1)$-dimensional faces in any pulling triangulation of $P_{A_n^+} \cap \partial \mathcal{L}_n$ is*

$$f_{j-1}\left(\triangle_<(P_{A_n^+} \cap \partial \mathcal{L}_n)\right) = \frac{1}{j+1}\binom{n+j}{j}\binom{n}{j}.$$

Proof uses revlex order.

**Note:**

$$P_n^{(0,0)}(x) \quad = \quad \sum_{j=0}^{n} \binom{n+j}{n}\binom{2j}{j}\left(\frac{x-1}{2}\right)^j$$

whereas

$$\begin{aligned} &\sum_{j=0}^{n} \frac{1}{j+1}\binom{n+j}{j}\binom{n}{j}\left(\frac{x-1}{2}\right)^j \\ = \quad &\sum_{j=0}^{n} \binom{n+j}{n} C_j \left(\frac{x-1}{2}\right)^j \end{aligned}$$

The relation between the root polytope $P_{A_n^+}$ and the Legendre polytope $\mathcal{L}_n$ is thus a "geometric enhancement" of the relation between the Catalan numbers and central binomial coefficients.

## Delannoy numbers in the "Legendrotope"

**Theorem 9** *For* $i > 0$ *the Delannoy number* $d_{n,n-i}$ *is the number of all faces* $F$ *in the lexicographic pulling triangulation of* $\partial\mathcal{L}_n$ *that contain at least one point in the generalized orthant* $\{(x_0, \ldots, x_n) \,:\, x_0 < 0, x_1 < 0, \ldots, x_i < 0\}$.

This is true because:

**Theorem 10** *The number of those* $(k+i-1)$*-dimensional faces in the lexicographic pulling triangulation of* $\partial\mathcal{L}_n$ *which contain at least one vertex of the form* $e_s - e_t$ *for each* $t \in \{0, 1, \ldots, i-1\}$ *is* $\binom{n+k}{k+i}\binom{n-i}{k}$.

## “Peek under the hood:”

Obviously,

$$d_{n,n-i} = \sum_{k=0}^{n-i} \binom{n+k}{k+i} \binom{n-i}{k}.$$

Exotically,

$$\begin{aligned}
&\binom{n+k}{k+i}\binom{n-i}{k} \\
&= \sum_{u=1}^{n-i+1} \binom{n-i+1}{u} \binom{k+i-1}{k+i-u} \sum_{v} \binom{n-i+1-u}{v} \binom{u-1}{k-v} \\
&= \sum_{u,v} \binom{n-i+1}{u, v, n-i+1-u-v} \binom{k+i-1}{k+i-u, k-v, u+v-k-1}.
\end{aligned}$$

# Part III: Lattice path enumeration

## Shifted Legendre polynomials

$$\widetilde{P}_n(x) := P_n(2x-1).$$

Easy to show:

$$\widetilde{P}_n(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n+k}{n-k} \binom{2k}{k} x^k.$$

**Definition 4** *Let $u$, $v$, $w$ be commuting variables. We define the* weighted Delannoy numbers $d_{m,n}^{u,v,w}$ *as the total weight of lattice paths from $(0,0)$ to $(m,n)$, with steps $(0,1)$, $(1,0)$, and $(1,1)$, where each step $(0,1)$ has weight $u$, each step $(1,0)$ has weight $v$, and each step $(1,1)$ has weight $w$. The weight of a lattice path is the product of the weights of its steps.*

**Surprise!** (?)

$$d_{n,n}^{u,v,w} = (-w)^n \widetilde{P}_n\left(-\frac{uv}{w}\right)$$

Now

$$d_{n,n} = d_{n,n}^{1,1,1} = (-1)^n \widetilde{P}_n(-1) = (-1)^n P_n(-3) = P_n(3)$$

since $(-1)^n P_n(-x) = P_n(x) \ldots$

**Lattice path model for the shifted Legendre polynomials**

$$\widetilde{P}_n(x) = d_{n,n}^{1,x-1,1} = d_{n,n}^{1,x,-1}$$

Amusing detail:

$$S_{n+1,n+1}^{1,x,w} = \int_0^x d_{1,t,w}\ dt$$

is the total weight of lattice paths from $(0,0)$ to $(n+1, n+1)$ staying strictly below the line $(t,t)$ (see "royal paths", "weighted Schröder numbers").

## A combinatorial proof of the orthogonality of the Legendre polynomials

Assume $m < n$.

$$
\begin{aligned}
& (n+m+1)! \int_0^1 x^m \widetilde{P}_n(x)\, dx \\
= \quad & \sum_{k=0}^{n} (-1)^{n-k} \binom{n+k}{n-k} \binom{2k}{k} \frac{(n+m+1)!}{m+k+1}
\end{aligned}
$$

Weight the steps in the lattice paths with $1, 1, -1$. Number $r$, $c_1, \ldots, c_m$ and the $n$ rows with a permutation, such that the label of $r$ is less than the label of any $c_i$ and less than the label of any row containing a $(0,1)$ step. Cancel the diagonal steps with the $((1,0),(0,1))$ sequences, when possible. You will be left with pairs of lattice paths and permutations such that

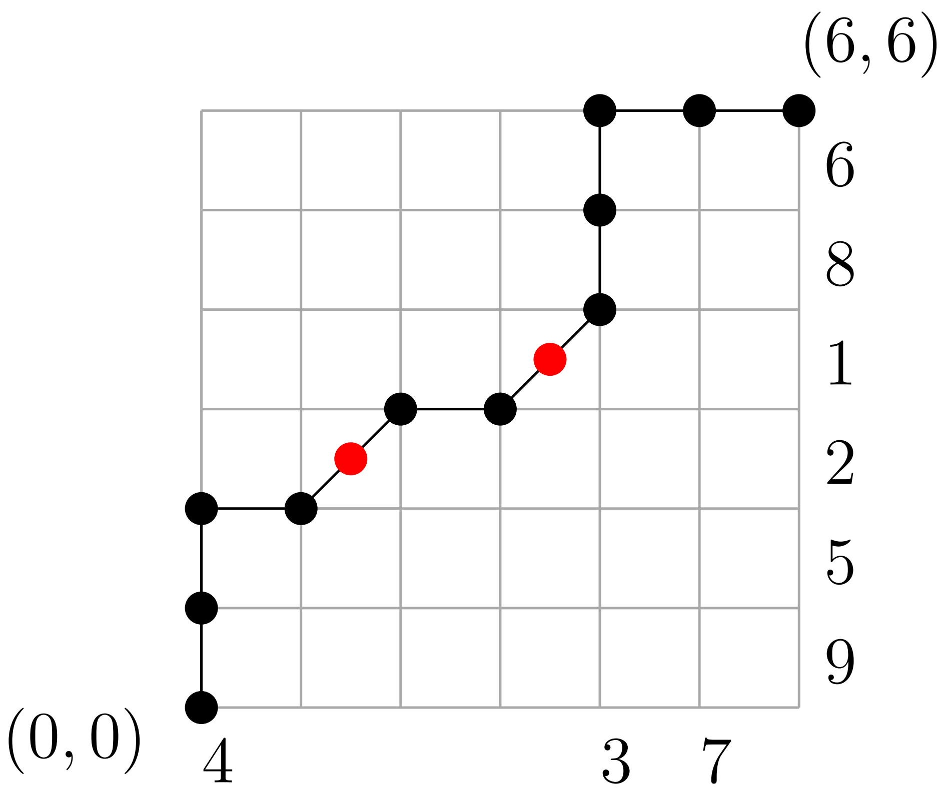
(a) $((1,0),(0,1))$ is forbidden;

(b) label of any row containing a diagonal step is less then the label of $r$

(b) makes the label of $r$ unique, (a) makes the lattice path depend on the position of the diagonal steps only ($\sim$ "rook placements").

**Example**

$n = 6$, $m = 2$.



## This also proves the orthogonality of Laguerre polynomials

Left to show:

$$p_n(m) := \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k}^2 (n-k)!(m+k)! = \delta_{m,n} \cdot n!$$

Here $(-1)^n p_n(-m) = (n+m-1)_n$ (by some certainly known coloring story).

## Going off-diagonal

Define *shifted Jacobi polynomials by*

$$\widetilde{P}_n^{(\alpha,\beta)}(x) := P_n^{(\alpha,\beta)}(2x-1).$$

Then we have

$$\widetilde{P}_n^{(0,\beta)}(x) = d_{n+\beta,n}^{1,x,-1}$$

As a consequence of the well known-relation $P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x)$ we get

$$\widetilde{P}_n^{(\alpha,0)}(x) = d_{n+\alpha,n}^{1,x-1,1}$$

**Corollary 4**

$$d_{n+\alpha,n} = \widetilde{P}^{(\alpha,0)}(2) = P_n^{(\alpha,0)}(3).$$

There is a combinatorial proof for $\beta \in \mathbb{Z}$ that the polynomials $\widetilde{P}_n^{(0,\beta)}(x)$ form an orthogonal basis with respect to the inner product

$$\langle f,g\rangle := \int_0^1 f(x)\cdot g(x)\cdot x^\beta \; dx$$

The proof merges into the proof of orthogonality of generalized Laguerre polynomials $L_n^{(\beta)}(x)$ with respect to

$$\langle f,g\rangle := \int_0^\infty f(x)\cdot g(x)\cdot x^\beta e^{-x} \; dx$$

# Concluding remarks

***Happy Birthday Lou!***

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