0.1. January 25, 2005. Consider Laplace’s equation $\Delta u = 0$ on a domain $E \subseteq \mathbb{R}^d$. Classically, solutions are harmonic functions. We are interested in the Dirichlet problem – to solve Laplace’s equation with the boundary condition $u = f$ on $\partial E$.

The probabilistic approach to the Dirichlet problem can be traced back to Friedrichs, Courant, and Levi. Suppose $E \subseteq \mathbb{Z}^d$. Let $u$ be a function defined on the vertices of $E$. We define a “Laplace” operator on such functions by

$$\Delta u(x) = u(x) - \frac{1}{2d} \sum_{y \sim x} u(y),$$

where $y \sim x$ means $y$ is a nearest neighbour of $x$.

Let $\xi$ be a path through $E$ starting at the point $x$. Let $\tau$ denote the first time that a path exits $E$. It is easy to see that the probability that the particular path $\xi$ is taken is

$$\Pi_x(\xi) = \left(\frac{1}{2d}\right)^\tau.$$

Then one can see that the Dirichlet problem with boundary function $f$ is solved explicitly by

$$u(x) = \sum_\xi \Pi_x(\xi)f(\xi_\tau) \equiv \Pi_x f(\xi_\tau).$$

By taking the lattice $\epsilon\mathbb{Z}^d$ and letting $\epsilon \to 0$ in a careful manner, we arrive at the probabilistic interpretation of the classical Dirichlet problem. The measure $\Pi_x$ becomes a measure on all continuous paths starting at $x$ – i.e. Weiner measure $\mathbb{W}$. The formulat $u(x) = \Pi_x f(\xi_\tau)$ still describes the solution to the Dirichlet problem.

We may consider a more general elliptic operator

$$L = \sum_{i,j=1}^d a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i \frac{\partial u}{\partial x_i},$$

where $\sum_{i,j=1}^d a_{ij}(x)\lambda_i\lambda_j > 0$ for all $x$, $\lambda_k \neq 0$. The probabilistic interpretation of the Dirichlet problem for $L$ involves not Brownian motion but a more general stochastic process, called an $L$-diffusion (which also has continuous sample paths).

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Date: March 28, 2005.
In general, Brownian motion and Wiener measure can be recovered from the transition density of the Laplacian, which is the normal density

\[ p(r, x; t, y) = \frac{1}{(2\pi(t-r))^{d/2}} e^{-|x-y|^2/2(t-r)}. \]

The same can be done for the \( L \)-diffusion, where the transition density is the fundamental solution of the \( L \)-heat equation \( \dot{u} + Lu = 0 \). An alternative is Itô’s approach, solving the stochastic differential equation \( d\xi_t = \sigma(\xi_t) dB_t + b(t) dt, \quad \xi_0 = x \), where \( \sigma \sigma^\top = a \).

We will also be interested in the semilinear operator \( u \mapsto Lu - \psi(u) \) (e.g. \( \psi(u) = u^\alpha, \alpha > 1 \)). The Dirichlet problem for this operator is \( Lu = \psi(u) \) in \( E \), \( u = f \) on \( \partial E \). There are a number of interesting phenomena associated to this problem.

1. All positive \( u \) have a common upper bounding function \( w \) (discovered by Keller-Ussuerman in 1957).
2. \( w \) is a maximal solution.
3. If \( \partial E \) is smooth, then \( Lu = \psi(u) \) in \( E \), \( u = \infty \) on \( \partial E \) admits a solution (in fact, \( w \) is such a solution).

**Exercise 0.1.** Show that with \( E = \mathbb{R}^d - \{0\} \), the equation \( \Delta u = u^\alpha \) admits solutions \( u(x) = q|x|^{-\alpha/(\alpha-1)} \) for some \( q > 0 \), if \( d < 2\alpha/(\alpha-1) \). Find which \( q \), and show there are no solutions if \( d \geq 2\alpha/(\alpha-1) \).

Recall that \( u(x) = \Pi_x f(\xi_t) \) solves \( Lu = 0 \) on \( E \), \( u = f \) on \( \partial E \), where \( (\xi_t, \Pi_x) \) is an \( L \)-diffusion. A similar role for the semilinear case is played by a superdiffusion. This is a measure valued process indexed not only by time but by subdomains \( D \) of the domain \( E \). \( \xi_t \) is replaced by an exit measure \( X_D \) on the boundary \( \partial D \) of a subdomain \( D \). \( f(\xi_t) \) is replaced by \( \langle f, X_D \rangle = \int f dX_D \). The initial state \( x \in D \) is replaced by a measure on \( D \), and so our process is \( \Pi_\mu \). The solution can then be expressed as

\[ u(x) = -\log \Pi_\delta e^{-\langle f, X_D \rangle}. \]

**0.2. January 27, 2005.** The solution to an equation of the form \( \Delta u = \psi(u) \) is given, probabilistically, in terms of a super-Brownian motion \( (X_D, \Pi_\mu) \). Roughly, we are thinking of a random cloud of particles (with density given by \( \mu \), and absorbing barrier described by \( X_D \)).

Super-Brownian motion can be described as a certain continuum limit of a discrete process, called a *Discrete Branching Exit Markov System*. In such a system, a finite number of particles begin following Brownian paths, but die at random times. (If the initial particles are at points \( x_1, \ldots, x_n \), then the initial probability measure is \( \mu = \delta_{x_1} + \cdots + \delta_{x_n} \).) When a particle dies, it spawns a finite (random) number of offspring. The process continues until some particles reach the boundary, at points \( y_1, \ldots, y_m \). The exit measure \( X_D \) is then \( X_D = \delta_{y_1} + \cdots + \delta_{y_m} \). Then we can compute directly that there is a function \( u \) such that

\[ \Pi_\mu e^{-\langle f, X_D \rangle} = e^{-\langle u, \mu \rangle}. \]

If we then give each particle a mass which is a positive-integer-multiple of a parameter \( \beta \), and require that the mean life-span of each particle is \( \beta \), and then let \( \beta \) tend to 0, we arrive at super-Brownian motion.

Using such super-Brownian motions, we can find solutions to the equation \( \Delta u = \psi(u), u|_{\partial E} = f \), but only for functions (related to) \( \psi(u) = u^\alpha \) for \( 1 < \alpha \leq 2 \).
0.3. January 27, 2005 (continued). We would like to find positive solutions of \(\Delta u = 0\) and \(\Delta u = \psi(u)\) in \(E\). There is much classical literature, for example, regarding positive harmonic functions in \(E = B = \{x \in \mathbb{R}^d; |x| < 1\}\). The typical classical tool is the Poisson integral: the general solution of \(\Delta u = 0\), \(u|_{\partial B} = f\) is given by

\[
u(x) = \int_{\partial B} k(x, y)f(y)dS(y),
\]

where \(S\) is surface measure on \(B\), and \(k\) is the Poisson kernel,

\[
k(x, y) = \frac{1 - |x|^2}{|x - y|^{d+2}}.
\]

Now, if \(\nu\) is any finite (positive) measure on \(\partial B\), then

\[
h_\nu(x) = \int_{\partial B} k(x, y)d\nu(y).
\]

In fact, there is a one-to-one correspondence between harmonic functions in \(B\) and positive measures on \(\partial B\), via the above formula. This fact extends quite widely, to any bounded region with smooth boundary, and any elliptic operator \(L\) in place of \(\Delta\). (Of course, the relevant kernel is no longer \(k\), and cannot in general be written down explicitly.)

The probabilistic interpretation of the above classical facts is quite simple. In the unit ball, the density of the exit measure for Brownian motion is precisely the Poisson kernel. In general, of \(\xi\) is a Brownian motion in a region \(E\), and \(k_D\) is the density of the exit measure, then for any measurable subset \(M \subseteq \partial E\),

\[
\Pi_x\{\xi_\tau \in M\} = \int_M k_D(x, y)dS(y).
\]

Our objective is to describe the set \(\mathcal{U}(E)\) of all positive solutions of \(Lu = \psi(u)\) in \(E\). As a first step, we wish to describe the set \(\mathcal{U}_1(E)\) of moderate solutions, which consists of all solutions \(u\) that are each bounded above by some harmonic function \(h\).

There is a one-to-one correspondence between \(\mathcal{U}_1(E)\) and a certain subset \(\mathcal{H}_1(E)\) of the set \(\mathcal{H}(E)\) of harmonic functions in \(E\): \(h\) corresponds to \(u\) if \(h\) is the minimal harmonic functions dominating \(u\). Equivalently (though non-trivially), that the correspondence assigns to \(h\) the maximal element of \(\mathcal{U}(E)\) dominated by \(h\). This allows each \(u\) to be labeled by a finite measure \(\nu\) if \(u = u_\nu\) corresponds to \(h_\nu\). However, since \(\mathcal{H}_1(E)\) is a strict subset of \(\mathcal{H}(E)\), not all finite positive measures \(\nu\) leads to a solution. Say that \(\nu \in \mathbb{N}_1\) if \(h_\nu \in \mathcal{H}_1(E)\).

Second step: the set \(\mathcal{U}_0(E)\) of \(\sigma\)-moderate solutions is defined by

\[
u(u) = \int \psi(u(x))d\nu(x)\]

\[
u(u) \leq \int \psi(u(x))d\nu(x)\]

If \(u \in \mathcal{U}_0(E)\), it corresponds to a measure \(\nu\), since \(u = \lim u_n\) and \(u_n\) is labeled by a unique measure \(\nu_n\); \(\nu\) is a the limit measure of \(\nu_n\). Such measures are said to be in the set \(\mathbb{N}_0\). Unfortunately, it is possible that \(u_\nu = u_{\nu'}\) for \(\nu, \nu' \in \mathbb{N}_0\) even if \(\nu \neq \nu'\).

The third step is a very nice characterization of all \(\sigma\)-moderate solutions by their boundary trace: \(\text{tr}u = (\Gamma, \nu)\), where \(\Gamma \subseteq \partial E\) and \(\nu\) is a \(\sigma\)-finite measure on \(\partial E - \Gamma\). \(\Gamma\) is the set of points of “rapid growth” of the function \(\psi'(|u(x)|)\). (Rapid growth is defined in terms of Brownian motion – the integral of the function along a Brownian path exiting in \(\Gamma\) is a.s. infinite.) The measure \(\nu\) is given by

\[
u(B) = \sup\{\mu(B); \mu \in \mathbb{N}_1, \mu(\Gamma) = 0, u_\mu \leq u\}.\]
1. DIFFUSIONS AND LINEAR PARTIAL DIFFERENTIAL EQUATIONS

1.1. February 1, 2005. A stochastic process is a model of a particle moving randomly through space. For the mathematical description, the ingredients are:

- two measurable spaces – the state space \((E, \mathcal{B})\) and the auxiliary space \((\Omega, \mathcal{F})\)
- the time set \(T \subseteq \mathbb{R}\) (usually \(T = \mathbb{R}_{+} = [0, \infty)\))
- a family \((\xi_{t}, \Pi)\) with \(\Pi\) a probability measure on \((\Omega, \mathcal{F})\), \(\xi_{t}(\omega) \in E, \omega \in \Omega, t \in T\), and \(\xi_{t}\) an \(\mathcal{F}\)-measurable map.

A Markov process is a family of stochastic processes \((\xi_{t}, \Pi_{x})\) indexed by all \(x \in E\), with the initial condition \(\Pi_{x}\{\xi_{0} = x\} = 1\). It also has the following structure:

- A filtration \(\mathcal{F}_{t}\) of \(\Omega\): for \(t > s \geq 0\), \(\mathcal{F}_{t} \subseteq \mathcal{F}\) and \(\mathcal{F}_{s} \subseteq \mathcal{F}_{t}\)
- Shift operators \(\theta_{s}: \Omega \rightarrow \Omega\) \(\mathcal{F}\)-measurable satisfying \(\theta_{s}\mathcal{F}_{t} \subseteq \mathcal{F}_{s+t}\)
- \(\xi_{s}(\omega)\) is \(\mathcal{F}_{t}\)-measurable for all \(t \geq s\)
- \(\theta_{s}\xi_{t}(\omega) \equiv \xi_{t}(\theta_{s}\omega) = \xi_{s+t}(\omega)\)
- The Markov property: For \(A \in \mathcal{F}_{s}\) and \(B \in \mathcal{F}\),

\[
\Pi_{x}(A \cap \theta_{t}B) = \int_{A} \Pi_{\xi_{t}(\omega)}(B) \Pi_{x}(d\omega).
\]

The Markov property implies that for \(Y \in \mathcal{F}_{t}\) and \(Z \in \mathcal{F}\),

\[
\Pi_{x}(Y \theta_{t}Z) = \Pi_{x}(Y \Pi_{\xi_{t},Z}),
\]

which is short-hand for

\[
\int_{\Omega} Y(\omega) \theta_{t}(\omega) Z \Pi_{x}(d\omega) = \int_{\Omega} Y(\omega) \int_{\Omega} Z(\tilde{\omega}) \Pi_{\xi_{t}(\omega)}(d\tilde{\omega}) \Pi_{x}(d\omega).
\]

(In general, we will write \(Y \in \mathcal{F}\) to mean \(Y\) is a positive \(\mathcal{F}\)-measurable function.)

The transition function associated to the Markov process is

\[
p_{t}(x, B) = \Pi_{x}\{\omega; \xi_{t}(\omega) \in B\}.
\]

It satisfies the following two properties:

- \(p_{t}(x, E) = 1\).
- \(\int_{E} p_{s}(x, dy)p_{t}(y, B) = p_{s+t}(x, B)\), for \(s, t \geq 0\), \(B \in \mathcal{B}\), and \(x \in E\).

We now proceed to construct a Markov process from a transition functions. Given the measurable space \((E, \mathcal{B})\), the canonical choice of \(\Omega\) is a space of paths \(\omega(t) \in E\) for \(t \geq 0\). \(\mathcal{F}\) is generated by \(\{\omega(t) \in B\}, t \geq 0, B \in \mathcal{B}\). \(\mathcal{F}_{t}\) is generated by \(\{\omega(s) \in B\}, s \in [0, t], B \in \mathcal{B}\). The shifts are defined by \(\theta_{s}\omega(t) = \omega(s + t)\).

Now, from our transition function \(p_{t}(x, B)\), we construct the following general function: for \(0 < t_{1} < t_{2} < \cdots < t_{n}\) and \(B_{1}, B_{2}, \ldots, B_{n} \in \mathcal{B}\),

\[
p(x; t_{1}, B_{1}; t_{2}, B_{2}; \ldots; t_{n}, B_{n}) \equiv \int_{B_{1}} \int_{B_{2}} \cdots \int_{B_{n}} p_{t_{1}}(x, dy_{1}) p_{t_{2} - t_{1}}(y_{1}, dy_{2}) \cdots p_{t_{n} - t_{n-1}}(y_{n-1}, dy_{n}).
\]

We then define the probability measures \(\Pi_{x}\) on \(\Omega\) via

\[
\Pi_{x}\{\omega(t_{1}) \in B_{1}, \ldots, \omega(t_{n}) \in B_{n}\} = p(x; t_{1}, B_{1}; \ldots; t_{n}, B_{n}).
\]

Defining the variables \(\xi_{t}\) by \(\xi_{t}(\omega) = \omega(t)\), we automatically get the formula

\[
\Pi_{x}\{\xi_{t_{1}}(\omega) \in B_{1}, \ldots, \xi_{t_{n}}(\omega) \in B_{n}\} = p(x; t_{1}, B_{1}; \ldots; t_{n}, B_{n}).
\]
Usually, we will deal with transition functions given by a transition density; i.e. a function \( p_t(x, y) \) satisfying

\[
\begin{align*}
\int_E p_s(x, y)p_t(y, z)dy &= p_{s+t}(x, z), \quad \text{for all} s, t > 0, x, z \in E \\
\int_E p_t(x, y)dy &= 1
\end{align*}
\]

then \( p_t(x, B) = \int_B p_t(x, y)dy \) is a transition function. By far the most important example is Brownian motion in \( \mathbb{R}^d \), whose transition density is given by a Gaussian

\[ p_t(x, y) = \frac{(2\pi t)^{-d/2}}{\pi^d} e^{-|x-y|^2/2t}. \]

More generally, we consider diffusion. Let

\[ L = \sum_{i,j} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b_i(x) \frac{\partial}{\partial x_i}, \]

where \( a_{ij}(x) \) is positive definite and \( a_{ij}(x), b_i(x) \) are Hölder continuous.

**Theorem 1.1.** There exists a unique continuous function \( p_t(x, y) \) s.t.

1. For every \( y, u(t, x) = p_t(x, y) \) satisfies \( \partial u/\partial t = Lu \) for \( t > 0 \).
2. \( p_t(x, y) > 0 \) and it is bounded on \( \{(x, y); t + |x - y| > \delta\} \) for any \( \delta > 0 \).
3. If \( \varphi \) is bounded and continuous, then \( \int_E p_t(x, y)\varphi(y)dy \to \phi(a) \) as \( t \downarrow 0, x \to a \).

This fundamental solution of the L-evolution equation is the transition density of a Markov process (which has a continuous version).

**1.2. February 3, 2005.** Recall, a Markov process has ingredients \( \xi = (\xi_t, \mathcal{F}_t, \theta_t) \) in state space \( (E, \mathcal{B}) \) (where \( x \in E \)) and auxiliary space \( (\Omega, \mathcal{F}) \). The Markov property is \( \Pi_x(Y\theta_t Z) = \Pi_x[Y(\Pi_t Z)] \) for \( Y \in \mathcal{F}_t, Z \in \mathcal{F} \), where we drop the \( \cap \) on the right. The transition function is \( p_t(x, B) = \Pi_x(\{\xi \in B\}) \) for \( B \in \mathcal{B} \). We usually consider those Markov processes whose transition function has a density, also called \( p_t \): \( p_t(x, dy) = p_t(x, y)dy \). The prime example is when \( p_t(x, y) = (2\pi t)^{-d/2} e^{-|x-y|^2/2t} \), for which the Markov process is \( d \)-dimensional Brownian motion.

Today we consider stopping times. Let \( \mathcal{F}_t \) be a filtration of \( (\Omega, \mathcal{F}) \). A random variable \( \tau(\omega) \) with values in \([0, \infty]\) is a stopping time if \( \{\tau \leq t\} \in \mathcal{F}_t \) for every \( t \in [0, \infty) \). (This says that \( \tau \) is independent of the future.) Associated to \( \tau \) is a \( \sigma \)-algebra, the Pre-\( \tau \) \( \sigma \)-algebra \( \mathcal{F}_\tau \), defined by

\[ C \in \mathcal{F}_\tau \text{ if } C \in \mathcal{F} \text{ and } C \cap \{\tau \leq t\} \in \mathcal{F}_t \text{ for all } t \geq 0. \]

An example, in the case of a path-space Markov process, is \( \tau(\omega) = \) the first time the path \( \omega \) exits some region.

A stopping time \( \tau \) is called simple if it takes at most a countable set of values. We can approximate any stopping time by simple ones in an appropriate fashion. Consider the diadic decomposition functions \( \varphi_n(t) = k/2^n \) for \( (k - 1)/2^n \leq t < k/2^n \). Note that \( \varphi_n(t) \downarrow t \), and so defining \( \tau_n = \varphi_n(\tau) \), we have \( \tau_n \downarrow \tau \).

**Definition 1.2.** A Markov process is called strong Markov (or is said to have the strong Markov property) if for every stopping time \( \tau \), \( Y \in \mathcal{F}_\tau \) (recall this means \( Y \) positive \( \mathcal{F}_\tau \)-measurable), and \( Z \in \mathcal{F} \),

\[ \Pi_x(Y 1_{\tau<\infty} \theta_\tau Z) = \Pi_x[Y 1_{\tau<\infty}(\Pi_\xi Z)]. \]

The \( 1_{\tau<\infty} \) is necessary since \( \tau \) can equal \( \infty \), while \( \xi_\infty \) is not defined. (To simplify notation, we could just agree that \( \xi_\infty \equiv 0 \).) Since \( \tau = \text{const} \). is a stopping time, the Markov property actually follows from the strong Markov property. We are interested in the converse, when it occurs.
Lemma 1.3. The strong Markov property holds for every simple stopping time.

Proof. It is easy to see that \( \{ \tau = t \} \in \mathcal{F}_\tau \). Hence, \( Y \mathbf{1}_{\tau = t} \in \mathcal{F}_t \), and so by the Markov property,

\[
\Pi_x Y \mathbf{1}_{\tau = t} \theta_t Z = \Pi_x (Y \mathbf{1}_{\tau = t} \Pi_{\xi_t} Z) = \Pi_x (Y \mathbf{1}_{\tau = t} \Pi_{\xi_t} Z).
\]

The statement now follows by taking countable sums. □

Definition 1.4. Let \( p_t(x, B) \) be the transition function of a Markov process. Consider the semigroup \( T_t f(x) = \int_E p_t(x, dy) f(y) \). A Markov process is called Feller if \( T_t \) preserves the space \( C(E) \) of bounded continuous functions.

Should check that this is equivalent to the measure \( p_t(x, dy) \) being weak-* continuous in \( x \).

Lemma 1.5. Put \( Z \in Q \) if \( Z = f_1(\xi_{t_1}) \cdots f_n(\xi_{t_n}) \) for some \( 0 < t_1 < \cdots < t_n \) and \( f_1, \ldots, f_n \in C(E) \). If \( \xi \) is a Feller process, then \( \Pi_x Z \) is continuous for every \( Z \in Q \).

Proof. For the case \( n = 1 \), note that \( \Pi_x f_1(\xi_{t_1}) = \int_E p_t(x, dy) f_1(y) = T_t f_1(x) \), and this is continuous by the definition of a Feller process. Proceeding by induction, note that \( f_n(\xi_{t_n}) = \theta_{t_{n-1}} f_n(\xi_{t_n-t_{n-1}}) \)

By the Markov property, \( \Pi_x Z = \Pi_x \tilde{Z} \), where \( \tilde{Z} = f_1(\xi_{t_1}) \cdots f_{n-1}(\xi_{t_{n-1}}) \Pi_{\xi_{t_{n-1}}} f_n(\xi_{t_n-t_{n-1}}) \). By before, \( \Pi_{\xi_{t_{n-1}}} f_n(\xi_{t_n-t_{n-1}}) \cdot f_{n-1}(\xi_{t_{n-1}}) \) is continuous in \( \xi_{t_{n-1}} \), so by inductive assumption \( \Pi_x Z \) is continuous. □

Theorem 1.6 (Dynkin, Yuschevich, 1956). Every right-continuous Feller process is strong Markov.

This follows from the Multiplicative systems theorem:

Theorem 1.7 (Dynkin). Let a linear space of bounded functions \( \mathcal{H} \) contain 1 and be closed under bounded convergence. If \( \mathcal{H} \) contains a multiplicative family \( Q \), then it contains all bounded \( \sigma(Q) \)-measurable functions.

1.3. February 8, 2005. The following theorem was proved independently by Dynkin and Keny in 1952.

Theorem 1.8. If \( E \) is a metric space and if \( \sup_x p_t(x, U_\epsilon(x)^c) = o(t) \) for every \( \epsilon \)-neighbourhood of \( x \), then the process (whose transition function is \( p_t \)) is continuous.

Blumenthal’s 0-1 law.

Let \( \mathcal{F}_{0+} = \bigcap_{s>0} \mathcal{F}_s \).

Theorem 1.9 (Blumenthal’s 0-1 law). Suppose \( \mathcal{F} \) is the Kolmogorov \( \sigma \)-algebra and \( \xi_t \) is right-continuous. If \( A \in \mathcal{F}_{0+} \), then for every \( x \), \( \Pi_x (A) \in \{ 0, 1 \} \).

Proof. Since \( Y \in \mathcal{F}_{0+} \), \( Y \in \mathcal{F}_s \) for all \( s > 0 \). Using the Markov property,

\[
\Pi_x Y \theta_s Z = \Pi_x Y \Pi_x \xi_s Z.
\]

Now, we use the same multiplicative theorem \( Q \) as above: \( Z \in Q \) iff \( Z = f_1(\xi_{t_1}) \cdots f_n(\xi_{t_n}) \) for some \( f_j \in C(E) \). Now, \( \theta_s Z \to Z \) and \( \Pi_x Z \to \Pi_{\xi_t} Z \) as \( s \downarrow 0 \). If \( Y \) is bounded, then by the dominated convergence theorem, from the Lemma above we have

\[
\Pi_x Y Z = \Pi_x (Y \Pi_{\xi_t} Z) = \Pi_x Y \Pi_x Z,
\]

for \( Z \in Q \). Applying the multiplicative systems theorem the same relation holds for all \( Z \in \mathcal{F} \) bounded. In particular, taking \( Y = Z = \mathbf{1}_A \) for \( A \in F_{0+} \), we get \( \Pi_x (A) = \Pi_x (A)^2 \), which completes the proof. □
First exit times.

Let $\mathcal{D}$ be a domain, and define the first exit time from $\mathcal{D}$ to be
\[ \tau_\mathcal{D} = \inf\{t > 0 ; \xi_t \notin \mathcal{D}\}. \]

Lemma 1.10. If $\xi$ is a continuous process in a metric space $E$, then for every open $\mathcal{D}$, $\tau_\mathcal{D}$ is a stopping time and $\{\tau_\mathcal{D} = 0\} \in \mathcal{F}_0$. Hence, by Blumenthal’s 0-1 law, $\Pi_x\{\tau_\mathcal{D} = 0\}$ is either 0 or 1 for each $x$.

Proof. Let $\varphi(x) = \text{dist}(x, \mathcal{D}^c)$, a continuous functions. Consider
\[ A_t^r = \{\xi_s \notin \mathcal{D} \text{ for some } s \in [r, t]\}. \]

In fact,
\[ A_t^r = \{\min_{s \in [r, t]} \varphi(x)_{is} = 0\} = \bigcup_{s \in [r, t]} \{\varphi(x) = 0\} \in \mathcal{F}_t. \]
Well, $\{\tau_\mathcal{D} \leq t\} = \bigcup_n A_t^{1/n} \in \mathcal{F}_t$, so $\tau_\mathcal{D}$ is a stopping time. Also, $\{\tau_\mathcal{D} = 0\} = \bigcap_n \{\tau_D \leq 1/n\} \in \mathcal{F}_0$.

2. Brownian Motion and Harmonic Functions

Brownian motion is a Markov process with transition function
\[ p_t(x, dy) = (2\pi t)^{-d/2} e^{-|x-y|^2/2t} dy. \]
First, note that
\[ p_t(x, U_{c}(x)^c) = \int_{|y-x|>\epsilon} (2\pi t)^{-d/2} e^{-|x-y|^2/2t} dy = (2\pi t)^{-d/2} \int_{r>\epsilon/\sqrt{t}} e^{-r^2/2t}dr. \]
It can be shown that this is $o(t)$. The same sort of argument shows that any diffusion is a continuous process.

Recall the canonical process can be defined with an auxiliary space $(\Omega, \mathcal{F})$ with $\Omega$ a space of continuous paths, and $\mathcal{F}$ the Kolmogorov $\sigma$-algebra. A Markov process can then be recovered from its transition function via a limiting process from
\[ \Pi_x\{\xi_{t_1} \in B_1, \ldots, \xi_{t_n} \in B_n\} = \int_{B_1 \times \cdots \times B_n} p_{t_1}(x, y_1)p_{t_2-t_1}(y_1, y_2) \cdots p_{t_n-t_{n-1}}(y_{n-1}, y_n)dy_1 \cdots dy_n. \]
Note that $\Pi_x\{\xi_t - \xi_s \in B\} = \int_B p_{t-s}(y)dy$, where $p_t(y) = (2\pi t)^{-d/2} e^{-|y|^2/2t}$. For Brownian motion, $\Pi_x\{\xi_t - \xi_s\} = 0$ and $\Pi_x|\xi_t - \xi_s|^2 = t-s$, one can compute readily that
\[ \Pi_x\{\xi_{t_1} - \xi_{t_0} \in B_1, \ldots, \xi_{t_n} - \xi_{t_{n-1}} \in B_n\} = \prod_{k=1}^{n} \Pi_x\{\xi_{t_k} - \xi_{t_{k-1}} \in B_k\}. \]
Thus, Brownian motion has independent increments. We can formulate this as part of the following theorem:

Theorem 2.1. Brownian motion can be uniquely characterized as a family $(\xi_t, \Pi_x)$ in a canonical space with independent increments such that $\eta_{s,t} = \xi_t - \xi_s$ is normal with $\Pi_x|\eta_{s,t}|^2 = t-s$, and $\xi_0 = x$.

Corollary 2.2. $(\xi_t, \Pi_0) = (\xi_t + x, \Pi_x)$. This follows simply by setting $\xi_t = \xi_t + x$. Then $\xi_0 = x$ and $\xi_t - \xi_s = \xi_t - \xi_s$. The corollary now follows from the theorem.
Corollary 2.3. Let $S$ be a rotation about 0. Then $(\xi_t, \Pi_0) = (S\xi_t, \Pi_0)$.

Theorem 2.4 (Self-similarity). Put $\xi_t = \xi_{\lambda^2 t}/\lambda$. Then $(\xi_t, \Pi_0) = (\xi_t, \Pi_0)$. (That is, they have the same probability distribution.)

Proof. $\xi_0 = 0$, and $\eta_{s,t} = \xi_t - \xi_s = (\xi_{\lambda^2 t} - \xi_{\lambda^2 s})/\lambda$. We need to check the parameters of this normal density.

$$
\Pi_x \eta_{s,t} = \frac{1}{\lambda} \Pi_x (\xi_{\lambda^2 t} - \xi_{\lambda^2 s}) = 0.
$$

$$
\Pi_x |\eta_{s,t}|^2 = \frac{1}{\lambda^2} \Pi_x (\xi_{\lambda^2 t} - \xi_{\lambda^2 s})^2 = \frac{1}{\lambda^2} (\lambda^2 t - \lambda^2 s) = t - s.
$$

2.1. February 10, 2005. We can apply these regularity properties to exit times and exit points. Consider a ball $D = \{|x| < r\}$. Let $B \subseteq \partial D$, and let $A_u = \{ \omega(t) \in D \text{ for } 0 < t < u, w(u) \in B \}$. Let $A = \bigcup_{u>0} A_u$. then

$$
\Pi_0 \{ \tau_D \in B \} = \Pi_0 \{ \xi_0 \in A \}.
$$

Let $\bar{\xi}_t = S\xi_t$. Then we have

$$
\Pi_0 \{ \tau_D \in B \} = \Pi_0 \bigcup_{u>0} \{ \bar{\xi}_t \in D \text{ for } 0 < t < u, \bar{\xi}_u \in B \}
$$

$$
= \Pi_0 \bigcup_{u>0} \{ \xi_t \in D \text{ for } 0 < t < u, \xi_u \in S^{-1}B \}
$$

$$
= \Pi_0 \{ \tau_D \in S^{-1}B \}.
$$

Here, we’ve used the fact that $S^{-1}D = D$, which is obvious, and the fact that $A$ is in the Kolmogorov $\sigma$-algebra (not so obvious, but can be checked). This shows that the exit measure is uniform.

A similar calculation shows that, if $D_r$ is the ball of radius $r$, then $\Pi_0 \tau_{D_r} = (1/\lambda^2) \Pi_0 \tau_{D_{\lambda r}}$. In particular, $\Pi_0 \tau_{D_1} = C\lambda^2$ where $C = \Pi_0 \tau_{D_1}$. It follows that $\Pi_0 \tau_D$ is finite for any bounded domain $D$. (Of course, we must show that $C$ is finite; this is done in [D1].)

Now, suppose $D$ is a domain such that $\lambda D = D$ for all $\lambda > 0$ (so $D$ is a cone). Then we have

$$
\Pi_0 \{ \tau_D < t \} = \Pi_0 \{ \frac{1}{\lambda^2} \tau_D < t \} = \Pi_0 \{ \tau_D < \lambda^2 t \}.
$$

Letting $\lambda \uparrow \infty$ and $t \downarrow 0$ (independently), we see that $\Pi_0 \{ \tau_D = 0 \} = \Pi_0 \{ \tau_D < \infty \}$. From this, it follows that $\Pi_0 \{ \tau_D = 0 \text{ or } \infty \} = 1$. Moreover, we can compute hitting times. Let $C = D'$. Let $\sigma_C = \inf\{ t > 0 : \xi_t \in C \} = \tau_D$. Then $\{ \sigma_C = 0 \} \in \mathcal{F}_0$. By Blumenthal’s 0-1 law, we have either $\Pi_0 \{ \sigma_C = 0 \} = 1$ or $\Pi_0 \{ \sigma_C = \infty \} = 1$. Take the example when $C = [0, \infty)$. The latter case is impossible, since (by rotation invariance and continuity) it would follow that $\xi_t$ stays in either the upper or lower halfplane. This is a contradiction since, at $t = 1$, the probability of being at $(0, 1)$ is equal to the probability that it is at $(0, -1)$. Hence, we have $\Pi_0 \{ \sigma_C = 0 \} = 1$.

Exercise 2.5. Show that $\Pi_0 \{ \sigma_{[0, \epsilon]} = 0 \} = 1$ for all $\epsilon > 0$.

Exercise 2.6. Let $T_{a,b}$ be a triangle in $\mathbb{R}^3$ spanned by the points $0$, $a$, and $b$. Show that

$$
\Pi_0 \{ \sigma_{T_{a,b}} = 0 \} = 1.
$$
We now move on to consider Harmonic functions. Let $D$ be a domain in $\mathbb{R}^d$, $f \in C^2(D)$ is harmonic if $\Delta f = 0$. Let $B$ be a ball, $S$ the normalized Lebesgue measure on $B$, and $\gamma$ normalized surface measure on $\partial B$. Define the mean values of $f$ on $B$ and $\partial B$ by

$$I_B(f) = \int_B f(y) S(dy)$$
$$I_{\partial B}(f) = \int_{\partial B} f(y) \gamma(dy).$$

**Theorem 2.7** (MVP). If $f$ is harmonic in $D$, and $D$ contains a ball $B$ centred at $x$, then

$$I_B(f) = I_{\partial B}(f) = f(x).$$

(For the best proof, see Gilberg and Troudinger, “Elliptic Partial Differential Equations of Second Order,” 1998 edition. This is Theorem 2.1.)

**Theorem 2.8.** Every $f \in L^1_{loc}(D)$ with the MVP is $C^2(D)$ and satisfies $\Delta f = 0$ in $D$.

**Proof sketch:** Continuity follows from the solid mean value property $I_B f = f(x)$, by shifting the ball a small bit. Now, if $g \in C^2(D)$, then

$$\int_{|z-x|<\rho} f(z)g(|z-x|^2)dz = \int_{|y|<\rho} f(x+y)g(|y|^2)dy = c(\rho)f(x) \int_{0}^{\rho} g(r^2)dr,$$

the last equality following by changing to Polar coordinates, then applying the mean value property. Now, if $C = \{ z ; |z-x| = \rho \}$ then $I_C(f) = f(x) + \rho^2 \Delta f(x) + o(\rho^2)$. The proof now follows.

**Theorem 2.9** (Strong Maximum Principle). A harmonic function in $D$ cannot assume an interior maximizer (or minimizer) unless it is constant.

**Proof.** Suppose $f(x_0) \geq f(x)$ for $x_0 \in D$ and for all $x \in B$, where $B$ is a closed ball in $D$. Then $f - f(x_0) = \tilde{f} \geq 0$ in $B$. Hence $I_B(\tilde{f}) = \tilde{f}(x_0) = 0$. Consequently, $\tilde{f} = 0$ in $B$. Since $D$ is connected, it follows that $f = f(x_0)$ in $D$. 


**Lemma 2.10.** Let $D$ be a bounded domain, and let $\varphi$ be a bounded Borel function on $\partial D$. Then $f(x) = \Pi_x \varphi(\xi_{\tau_D})$ is harmonic in $D$.

**Proof.** Let $C$ be a closed ball contained in $D$, and let $B \subseteq \partial C$. Let $\tau = \tau_D$ and $\tau' = \tau_B$. Then, by continuity, $\tau' < t < \infty$. Also, note that $\theta_{\tau'} \tau = \tau$. So,

$$f(x) = \Pi_x \varphi(\xi_{\tau}) = \Pi_x \varphi(\xi_{\theta_{\tau'} \tau}) = \Pi_x \theta_{\tau'} \varphi(\xi_{\tau}).$$

Now, using the strong Markov property, this becomes

$$\Pi_x \Pi_{\xi_{\tau'}} \varphi(\xi_{\tau}) = \Pi_x f(\xi_{\tau'}) = I_C f.$$

Hence, $f$ satisfies the solid mean value property, so it is harmonic. 

Recall the Dirichlet problem: If $D$ is a bounded domain, and $\varphi$ is a bounded Borel function on $\partial D$, does there exist a unique function $f$ satisfying $\Delta f = 0$ in $D$ and $f(x) \to \varphi(a)$ as $x \to a$, $x \in D$? The uniqueness follows easily from the maximum principle.

**Definition 2.11.** If $D$ is a bounded domain, call $a \in \partial D$ a regular point if $\Pi_a \{ \tau_D = 0 \} = 1$. 

An example of a non-regular point is the centre of a punctured disc. A sufficient condition for a point \( a \) to be regular is: there exists a \( d-1 \)-dimensional simplex \( A \) not overlapping with \( D \), having a vertex at \( a \). (Note, this is a lot weaker than an exterior cone condition – a domain in 2-dimensions with an arbitrarily sharp internal cusp is still regular at the cusp!) This follows trivially from Exercise 2.6.

**Lemma 2.12.** \( f(x) = \Pi_x \varphi(\xi_{\tau_D}) \rightarrow \varphi(a) \) as \( x \rightarrow a \) if \( a \) is a regular point, \( \varphi \) is continuous at \( a \), and \( \varphi \) is bounded.

**Proof.** We have \( \Pi_a \{ \tau_D = 0 \} = 1 \). Hence, \( \Pi_x \{ \tau_D > h \} \rightarrow 0 \) as \( x \rightarrow a \), \( x \in D \) for every \( h > 0 \). Roughly speaking: since \( \tau_D \) is small, \( \xi \tau \) is close to \( x \) and hence is close to \( a \). This means that \( \varphi(\xi_{\tau}) \) is close to \( \varphi(a) \), and so \( \Pi_x \varphi(\xi_{\tau}) \) is close to \( \varphi(a) \). For the details, see Chapter 2, Section 5, in [D3]. \( \square \)

Call a domain regular if every point in its boundary is regular. We have thus proved the following theorem.

**Theorem 2.13.** Let \( D \) be a bounded regular domain, and \( \varphi \) a continuous function on \( \partial D \). The Dirichlet problem \( \Delta f = 0 \) in \( D \), \( f = \varphi \) on \( \partial D \), is solved uniquely by

\[
f(x) = \Pi_x \varphi(\xi_{\tau_D}).
\]

There is a more general mean value property, using a harmonic measure rather than the uniform measure.

**Theorem 2.14.** Let \( U \) be a regular, relatively compact subset of \( D \). Then a harmonic function \( f \) in \( D \) satisfies the mean value property:

\[
f(x) = \Pi_x f(\xi_{\tau_U}) = \int_{\partial U} \Pi_U(x,dy)f(y).
\]

This follows immediately from the uniqueness of the solution to the Dirichlet problem.

3. Diffusions, and their Applications to Linear PDEs

We will begin by construction a diffusion starting from a transition function. We consider an elliptic differential operator:

\[
L = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x) \frac{\partial}{\partial x_i},
\]

where \( [a_{ij}(x)] \) is positive definite for each \( x \), \( a_{ij} \) and \( b_i \) are bounded and Hölder continuous. Let \( p_t \) be the heat kernel for \( L \) – i.e. \( e^{-tL}f = p_t * f \). \( p_t \) is uniquely determined by the properties

- \( \partial p_t / \partial t = L p_t \).
- \( p_t(x,y) > 0 \) and bounded on \( \{ t + |x - y| > \delta \} \) for \( \delta > 0 \).
- If \( f \) is a bounded continuous function, then \( \int_{\mathbb{R}^d} p_t(x,y)f(y)dy \rightarrow f(a) \) as \( t \rightarrow 0 \), \( x \rightarrow a \).

This heat kernel satisfies, for sufficiently small \( \beta > 0 \),

\[
p_t(x,y) \leq C(\beta, T) t^{-d/2} e^{-\beta |x-y|^2 / 2t},
\]

for all \( 0 < t < T \), \( x, y \in \mathbb{R}^d \). Using this, we may construct a Markov process in exactly the same manner we constructed Brownian motion. We will not harp in this, but rather will consider Itô’s approach: stochastic integration theory.
**Definition 3.1.** A martingale \( M_t(\omega) \) with respect to a filtration \( \mathcal{F}_t \) of \( \mathcal{F} \) is a stochastic process \((\Omega, \mathcal{F}, \Pi)\) satisfying \( \int_A M_s d\Pi = \int_A M_t d\Pi \) for all \( 0 \leq s < t \) and \( A \in \mathcal{F}_s \), and \( \mathbb{E}\{M_t|\mathcal{F}_s\} = M_s \) a.s. for \( s < t \).

Here we take the point of view that the measure \( \Pi \) is fixed, and we will reach starting points other than 0 by considering families \((\xi_t^s, \Pi)\) of stochastic processes indexed by points \( x \).

### 3.1. February 17, 2005

Recall, given \((\Omega, \mathcal{F}, \Pi)\) and a filtration \( \mathcal{F}_t \), \( M_t \) is a **martingale** if

1. \( M_t \) is adapted to \( \mathcal{F}_t \)
2. \( M_t \) is integrable
3. For every \( 0 \leq s < t \), \( \mathbb{E}(M_t|\mathcal{F}_s) = M_s \) a.s., which is to say that \( \int_A M_t d\Pi = \int_A M_s d\Pi \) for \( A \in \mathcal{F}_s \).

We can think of ordinary differential equations as integral equations:

\[
\begin{aligned}
    \left\{ \begin{array}{l}
        d\xi_t = b(\xi_t)dt \\
        \xi_0 = x
    \end{array} \right. \iff \xi_t = x + \int_0^t b(\xi_s)ds.
\end{aligned}
\]

Itô’s idea is to introduce a stochastic term in such an equation. Let \( B_t \) denote Brownian motion.

\[
\begin{aligned}
    \left\{ \begin{array}{l}
        d\xi_t = \sigma(\xi_t)dB_t + b(\xi_t)dt \\
        \xi_0 = x
    \end{array} \right. \iff \xi_t = \xi_x + \int_0^t \sigma(\xi_s)dB_s + \int_0^t b(\xi_s)ds.
\end{aligned}
\]

Here \( \sigma \) is a function \( \mathbb{R}^d \to \mathbb{R} \), \( b \) is a function \( \mathbb{R} \to \mathbb{R} \). To understand this, we need to make sense of the **stochastic integral**,

\[
M_t = \int_0^t Y_s dB_s.
\]

We define this through a class of “simple processes” \( \mathcal{P}_0 \), consisting of \( Y \) satisfying \( Y_s = F_u \mathbb{1}_{s>u} \) where \( F_u \in \mathcal{F}_u \), and \( \mathbb{E}F_u^2 < \infty \). We then define the integral by

\[
M_t = \int_0^t F_u \mathbb{1}_{s>u} dB_s \equiv F_u(B_t - B_{t\wedge u}).
\]

One can check easily that \( M_t \) is a continuous martingale. Moreover, if \( Y, \tilde{Y} \in \mathcal{P}_0 \), then

\[
\mathbb{E}M_t^2 = \int_0^t \mathbb{E}(Y_s^2)ds, \quad \text{and} \quad \mathbb{E}M_t \tilde{M}_t = \int_0^t \mathbb{E}(Y_s \tilde{Y}_s)ds.
\]

So now we have this stochastic integral \( \mathbb{I} \) mapping \( \mathcal{P}_0 \) into the space of continuous martingales. We can extend \( \mathbb{I} \) to the span of \( \mathcal{P}_0 \) by linearity, and then to the \( L^2 \)-closure of \( \mathcal{P}_0 \), which we refer to as \( \mathcal{P} \). To be precise, if \( M^{(n)} \) and \( M \) are continuous martingales, say \( M^{(n)} \to M \) if \( \mathbb{E}(M_t^{(n)} - M_t)^2 \to 0 \) for every \( t \). Say that \( Y^{(n)} \to Y \) if \( \int_0^t \mathbb{E}(Y_s^{(n)} - Y_s)^2ds \to 0 \). It is easy to see that the above identities for \( \mathbb{E}M_t^2 \) and \( \mathbb{E}M_t \tilde{M}_t \) holds for \( Y, \tilde{Y} \in \mathcal{P} \).

We can describe \( \mathcal{P} \) more efficiently. It consists of all \( Y \) such that:

1. \( Y_t(\omega) \) is measurable with respect to the \( \sigma \)-algebra in \([0, \infty) \times \Omega \) generated by the family \( \mathcal{P}_0 \)
2. \( \int_0^t \mathbb{E}Y_s^2ds < \infty \) for all \( t \).
Item 1 above classifies those processes which are predictable.

Denote by $\xi_t(r, x)$ the solution of the integral equation

$$\xi_t(r, x) = x + \int_r^t \sigma(\xi_s) dB_s + \int_r^t b(\xi_s) ds.$$  

(We say this is a solution if it satisfies the equation and is adapted to the Brownian motion.) Moreover, we may generalize by letting the initial point be random, but in the past: let $\eta \in \mathcal{F}_r$. That is, if $\xi_t(r, \eta)$, then $\xi_t(r, \eta)$ is this solution of

$$\xi_t = \eta + \int_r^t \sigma(\xi_s) dB_s + \int_r^t b(\xi_s) ds.$$

Now, let $0 < h < r < r$. Then, for now suppressing the non-stochastic term, we have

$$\xi_t(h, x) = x + \int_h^r \sigma(\xi_s) dB_s + \int_r^t \sigma(\xi_s) dB_s.$$  

Let $\eta = \xi_r(h, x) = x + \int_h^r \sigma(\xi_s) dB_s$. Note that $\eta \in \mathcal{F}_r$ (this is essentially by the definition of $\mathcal{F}_r = \mathcal{F}^r_r$, and so we have

$$\xi_t(h, x) = \xi_t(r, \xi_r(h, x)).$$

This is precisely the Markov property: the future, after $r$, does not depend on history up to time $r$. One can also show that the transition function $p(r, x; t, B) = \Pi\{\xi_t(r, x) \in B\}$ has the form $p(r, x; t, B) = p_{t-r}(x, B)$ (this follows from the fact that $B_s$ may be replaced by $B_s - B_r$ in the stochastic differential equation, since $B_s - B_r$ is a Brownian motion starting at 0 at time $r$). From here, we will find that $T_t f(x) = \int p_t(x, dy)f(y)$ is a semigroup. (Note, $T_t$ is equivalently defined as $T_t f(x) = \Pi f(\xi_t(0, x))$.)

**Stochastic Differentials, and Itô’s Formula.** We consider the Martingale

$$M_t(\omega) = \sum_{i=1}^d \int_0^t a_i(s, \omega) dB^i_s + \int_0^t b(s, \omega) ds.$$  

Here $a_i \in \mathcal{P}$, $B^1, \ldots, B^d$ are independent Brownian motions, $b(s, \omega)$ is adapted to the filtration $\mathcal{F}_s$, and $\int_0^t |b(s, \omega)| ds < \infty$. In this case, we write

$$dM_t = \sum_{i=1}^d a_i dB^i_t + b dt.$$  

We define a formal multiplication on stochastic differentials of this form, as follows:

$$(dB^i)^2 = dt, dB^i dB^j = dt dB^i = dB^i dy = (dt)^2 = 0 \text{ if } i \neq j,$$

and extending linearly. Using this definition, **Itô’s Formula** says that if $F_t = f(t, M^1_t, \ldots, M^\ell_t)$ are continuous Martingales with integral expressions, then $dF_t$ can be expressed through $dM^1, \ldots, dM^\ell$ by the formula

$$dF_t = f_0 dt + \sum_{i=1}^\ell f_i dM^i + \frac{1}{2} \sum_{i,j} f_{ij} dM^i \cdot dM^j.$$  

Here, $f_0 = \partial f / \partial t$, $f_i = \partial f / \partial x_i$, and $f_{ij} = \partial^2 f / \partial x_i \partial x_j$. 
3.2. **February 22, 2005.** Recall, we have a stochastic integral,

\[ M_t = \int_0^t Y_s dB_s, \]

where \( Y_s \) is a predictable process. (A sufficient condition for predictability is that \( Y_s \) is adapted to the Brownian filtration \( \mathcal{F}_s = \sigma\{B_r; r \leq s\} \), and is left-continuous.) In this case, \( M_t \) is a continuous martingale.

We have the stochastic differential equation,

\[ d\xi_t = \sigma(\xi_t)dB_t + b(\xi_t)dt, \quad t \geq r \]

\[ \xi_r = \eta \in \mathcal{F}_r, \]

where \( \sigma \) and \( b \) are Lipschitz. This, by definition, means that \( \xi_t \) satisfies the integral equation

\[ \xi_t = \eta + \int_r^t \sigma(\xi_s)dB_s + \int_r^t b(\xi_s)ds, \]

where \( \xi_t \) is automatically continuous and adapted to \( \mathcal{F}_t \).

Recall the semigroup \( T_t f(x) = \Pi f(\xi_0(0, x)) \). If \( f \in C(\mathbb{R}^d) \), then \( T_t f(x) \) is continuous (since \( T_t f(x) \to f(x) \) as \( t \to 0 \)).

Now we come back to Itô’s calculus. Given the stochastic integral equation

\[ \xi_t = \xi_0 + \int_0^t \sigma(\xi_s)dB_s + \int_0^t b(\xi_s)ds, \]

Itô’s formula allows us to determine the stochastic integral equation (in differential form) satisfied by \( f(\xi_t) \), where \( f \) is a smooth function. Writing the above equation in coordinates in differential form, we have

\[ d\xi_t^i = \sum_{k=1}^d \sigma_{ik}(\xi_t)dB^k_t + b_i(\xi_t)dt, \quad i = 1, 2, \ldots, d. \]

By letting \( \sigma_{ik}(t, \omega) = \sigma_{ik}(\xi_t(\omega)) \), and \( b_i(t, \omega) = b_i(\xi_t(\omega)) \), we can generalize this equation to

\[ dM^i = \sum_{k=1}^d \sigma_{ik}(t, \omega)dB^k + b_i(t, \omega)dt, \]

which is of course equivalent to

\[ M^i_t = M^i_0 + \int_0^t \sum_k \sigma_{ik}(s, \omega)dB^k_s + \int_0^t b_i(s, \omega)ds. \]

This integral equation makes sense so long as \( \sigma_{ik}(t, \omega) \) and \( b_i(t, \omega) \) are predictable and \( L^2_{loc} \).

**Theorem 3.2** (Itô). Suppose \( F_t = f(M^1_t, \ldots, M^\ell_t) \), where \( f_i = \partial f/\partial x_i \) and \( f_{ij} = \partial^2 f/\partial x_i \partial x_j \) are continuous and bounded. Then

\[ dF_t = \sum_{i=1}^\ell f_i dM^i_t + \frac{1}{2} \sum_{i,j=1}^\ell f_{ij} dM^i_t \cdot dM^j_t, \]

where \( dM^i_t \cdot dM^j_t \) is evaluated using the rules \( (dB^i_t)^2 = dt \), and \( dB^i_t dB^j_t = dtdB^i_t = dB^i_t dt = (dt)^2 = 0 \) for \( i \neq j \).
Applying this to the above stochastic differential equation, we get
\[ df(\xi_t) = df(\xi^1_t, \ldots, \xi^d_t) = \sum_i f_i d\xi^i_t + \frac{1}{2} \sum_{i,j} f_{ij} d\xi^i_t d\xi^j_t. \]
Well, \( d\xi^i_t d\xi^j_t = (\sum_k \sigma_{ik} \sigma_{jk}) dt = a_{ij} dt \). So, \( f(\xi_t) \) satisfies the stochastic differential equation
\[ df(\xi_t) = \left( \frac{1}{2} \sum_{i,j} a_{ij} f_{ij} + \sum_i b_i f_i \right) dt + \sum_{i,k} f_{i} \sigma_{ij} dB^k_t. \]
Denoting the elliptic operator \( f \mapsto \frac{1}{2} \sum a_{ij} f_{ij} + \sum b_i f_i \) as \( f \mapsto Lf \), and denoting \( \sum f_i \sigma_{ik} \) as \( g_k \), we can write this as the integral equation
\[ f(\xi_t) = f(\xi_0) + \int_0^t Lf(\xi_s) ds + \int_0^t \sum_k g_k dB^k_s. \]
The third term above is a continuous martingale. Hence, we have that \( f(\xi_t) - f(\xi_0) = \int_0^t Lf(\xi_s) ds \) is a continuous martingale. Taking expectations, starting the process at \( \xi_0 = x \), we have
\[ \mathbb{E} f(\xi^x_t) - f(x) - \mathbb{E} \int_0^t Lf(\xi^x_s) ds = 0, \]
since the expectation of the continuous Martingale is 0. Rewriting this using the definition of \( T_t \),
\[ T_t f(x) - f(x) - \int_0^t T_s Lf(x) ds = 0. \]
Hence,
\[ \frac{T_t f(x) - f(x)}{t} = \frac{1}{t} \int_0^t T_s Lf(x) ds \to Lf(x) \text{ as } s \to 0, \]
and \( (T_t f - f)/t \) is bounded in norm by \( \|Lf\| \). So we have \( T_t \) is a \( C_0 \)-semigroup with (weak)-infinitessimal generator \( L \). Hence it is the same semigroup as the one we constructed earlier using the generating function.

Now, suppose \( f_i \) and \( f_{ij} \) are bounded and continuous in \( E \). Then \( f \) is a solution of \( Lf = 0 \) in \( E \) iff \( \Pi_x f(\xi^x_{\tau_D}) = f(x) \) for all \( x \in \mathcal{D} \) compactly contained in \( E \) - we showed this before. But this also follows from Itô’s approach, by replacing \( t \) with \( \tau = \tau_{\mathcal{D}} \) in the above equation (which we can do, since stopping times preserve expectations):
\[ \mathbb{E} f(\xi^x_{\tau^x}) - f(x) = \mathbb{E} \int_{0}^{\tau} Lf(\xi^x_s) ds = 0, \]
and hence \( \Pi_x f(\xi_{\tau}) - f(x) = \Pi_x \int_0^\tau Lf(\xi_s) ds \). From this, it follows immediately that if \( Lf = 0 \) then \( \Pi_x f(\xi_{\tau}) = f(x) \). The other direction follows similarly.

This is the end of our treatment of Itô’s method.

**Perron’s Solution.**

Let \( E \) be a bounded domain, and \( \varphi \) a bounded Borel function on \( \partial E \). If \( (\xi_t, \Pi_x) \) is an \( L \)-diffusion, then we’ve seen that \( f(x) = \Pi_x \varphi(\xi^x_{\tau_E}) \) is a solution of \( Lf = 0 \) in \( E \).

We say that \( f \) is \( L \)-superharmonic (resp. \( L \)-subharmonic) if \( f(x) \leq \Pi_x \varphi(\xi^x_{\tau_E}) \) (resp. \( f(x) \geq \Pi_x \varphi(\xi^x_{\tau_E}) \). Let \( Q_+ \) be the class of function which are \( L \)-superharmonic and lower-semicontinuous in \( E \), and \( \liminf_{x \to z} f(x) \geq \varphi(z) \) for all \( z \in \partial E \); similarly, \( Q_- \) is the class of all \( L \)-subharmonic upper-semicontinuous functions in \( E \) satisfying \( \limsup_{x \to z} f(x) \leq \varphi(z) \) for all \( z \in \partial E \). Then \( q_+ = \inf Q_+ \)
is $L$-superharmonic and $q_- = \sup Q_-$ is $L$-subharmonic. It turns out that $q_+ = q_-$, and hence $q_+$ is the solution. The details can be found in [P].

3.3. February 24, 2005. Today we consider Green operators and Poisson operators. This material can be found in Chapter 2, Section 1 and Chapter 3, Section 1 of [D2], and in Chapter 6, Section 1 and Section 2 of [D1].

Green operators and Green functions.

Analytic results. Consider a bounded smooth ($C^{2,\lambda}$) domain $D$. (This means the boundary of $D$ is locally given by functions that are $C^2$ and whose second derivative is $\lambda$-Hölder continuous.) It is known that for every $f \in C^\lambda(D)$, there exists a unique solution of the problem $Lu = -f$ in $D$ and $u = 0$ on $\partial D$. This theorem can be proved using the Perron method.

If $u$ is such a solution, but $u = G_D f$. $G_D$ is the Green operator of $L$ in $D$. It has the following properties.

1. $G_D : C^\lambda(D) \to C^{2,\lambda}(D)$.
2. There exists a unique continuous function $g_D(x, y)$ from $D \times D$ to $[0, \infty]$ such that $G_D f(x) = \int_D g_D(x, y) f(y) dy$. This function is called the Green function of $L$ in $D$.
3. With $u_y(x) = g(x, y)$, we have $Lu = 0$ on $D \setminus \{y\}$ ($u_y$ blows up at $y$) and $u = 0$ on $\partial D$. In particular, $g_D(x, y) < \infty$ for $x \neq y$.
4. For all $x, y \in D$, $0 < g_D(x, y) < C\Gamma(x - y)$, where $C = C(L, D)$ and
   \[
   \Gamma(x) = \begin{cases}
   |x|^{2-d}, & \text{for } d \geq 3 \\
   (-\log |x|) \vee 1, & \text{for } d = 2 \\
   1, & \text{for } d = 1.
   \end{cases}
   \]

Probabilistic results. If $\xi$ is an $L$-diffusion, then
\[
G_D f(x) = \Pi_x \int_0^{r_D} f(\xi_s) ds.
\]

This operator can be extended beyond the smoothness conditions above. Consider a family of smooth domains $D_n$ exhausting $D$. Then we can see that $G_{D_n} f(x)$ is monotone increasing, and so $g_{D_n}(x, y)$ increases as well. Let $g_D(x, y)$ be the monotone limit. Note, $g_D$ may be uniformly $\infty$! Say that $D$ is a Greenean domain if, for every subdomain $\bar{D}$ compactly contained in $D$, the bound 4 above holds (with a constant $C(L, \bar{D})$ depending on the subdomain). Note, because $\Pi_x r_D < \infty$ of $D$ is bounded, we have that $G_D 1 = \Pi_x \int_0^{r_D} 1 ds = \Pi_x r_D < \infty$, and so the Green function is $< \infty$ a.e. Thus, all bounded domains are Greenean. (It can also be seen from the form of the Green function for the Laplace operator that all domains are $\Delta$-Greenean if $d \geq 3$.)

Poisson operator and Poisson kernel. Let $D$ be a bounded smooth domain, $\varphi \in C(\partial D)$. Denote by $K_D \varphi$ the unique solution of $Lu = 0$ in $D$, $u = \varphi$ on $\partial D$. $K_D$ is called the Poisson operator of $L$ in $D$.

Analytic results. There exists a continuous function $k_D(x, y)$ on $D \times \partial D$ such that
\[
K_D \varphi(x) = \int_{\partial D} k_D(x, y) \varphi(y) S(dy),
\]
where $S$ is normalized surface measure on $\partial D$. In the case $L = \Delta$, $D = \{|x| < 1\}$,
\[
k_D(x, y) = C \frac{1 - |x|^2}{|x - y|^q}.
\]
For a domain $\mathcal{D}$ in Euclidean space and $L = \Delta$, it is a fact that
\[ k_{\varphi}(x, y) = \frac{\partial g_{\varphi}(x, y)}{\partial n}, \]
the normal derivative at the boundary. (In greater generality, there is a relation between the Poisson kernel and the Green function, but it is not a normal derivative – the normality corresponds to the fact that the Riemannian metric corresponding to $\Delta$ is the Euclidean metric.)

Some properties of the Poisson kernel.
1. For every $y \in \partial \mathcal{D}$, $h_y(x) = k_{\varphi}(x, y)$ is a solution of $Lh_y = 0$ in $\mathcal{D}$, $h_y = 0$ on $\partial \mathcal{D} - \{y\}$.
2. For all $x \in \mathcal{D}, y \in \partial \mathcal{D}$,
\[ \frac{1}{C |x - y|^d} \leq k_{\varphi}(x, y) \leq C \frac{\rho(x)}{|x - y|^d}, \]
where $\rho(x) = \text{dist}(x, \partial \mathcal{D})$.

**Probabilistic results.** As we have already proved in various ways,
\[ K_{\varphi} \varphi(x) = \Pi_x \varphi(\xi_{\tau_{\mathcal{D}}}) \mathbb{1}_{\tau_{\mathcal{D}} < \infty}. \]
We also have that
\[ \Pi_x \{ \xi_{\tau_{\mathcal{D}}} \in B \} = \int_B k_{\varphi}(x, y) S(dy). \]
Analysts refer to this integral as $k_{\varphi}(x, B)$, a harmonic measure. Now, suppose that $U$ is compactly contained in $\mathcal{D}$. Then $K_U K_{\varphi} = K_{\varphi}$. (This follows from $\theta_{\tau_U} \xi_{\tau_{\mathcal{D}}} = \xi_{\tau_{\mathcal{D}}}$.)

Now, let $A = \partial \mathcal{D} \cap \partial U$ and $C = \partial U \cap \mathcal{D}$. Then
\[ k_{\varphi}(x, y) = k_U(x, y) \int_C k_U(x, z) S_U(dz) k_{\varphi}(y, z), \quad \text{for } x \in U, y \in A. \]
(This follows easily from the Markov property.) It follows similarly
\[ g_{\varphi}(x, y) = g_U(x, y) + \int_C k_U(x, z) S_U(dz) g_{\varphi}(z, y), \quad \text{for } x, y \in \mathcal{D}. \]


The material presented here can be found in [D2], Chapter 3, Section 1. Let $E$ be a domain, and let $\xi = (\xi_t, \Pi_x)$ be a Markov process. Let $\tilde{\xi}_t$ be the killed process at $\partial E$; that is,
\[ \tilde{\xi}_t(\omega) = \xi_t(\omega), \quad \text{for } 0 \leq t < \tau_E(\omega), \quad \tilde{\xi}_t(\omega) = \xi_{\tau_E}, \quad \text{for } t \geq \tau_E(\omega). \]
So we have $\tilde{\xi}_t, (\Pi_x)$, for $x \in E$. The filtration $\mathcal{F}_t$ of the process is generated by $\{\tilde{\xi}_s \in B\} = \{\xi_s \in B, s < \tau_E\}, s \leq t$.

The transition... [he erased too quickly]

Note, if $\partial E$ is smooth, the killed diffusion process corresponds to the operator $L$ with 0 boundary conditions.

**Killing with the rate** $\ell(x) \geq 0$. Again, given a Markov process $\xi_t$, the killed process $\tilde{\xi}_t$ with (random) killing time $\zeta$ has the conditional probability that $\zeta > t$ given by
\[ \alpha_t = e^{-\int_0^t \ell(\xi_s) ds}. \]
Suppose \( \ell \) is continuous. Then
\[
\frac{\alpha t + h}{\alpha t} = e^{-\int_{t}^{t+h} \ell(s)ds} = 1 - \int_{t}^{t+h} \ell(s)ds + o(h) = 1 - \ell(t)h + o(h).
\]
So, given a path \( \xi \) not killed until time \( t \), the conditional probability it is killed during \([t, t+h]\) is equal to \( \ell(t)h + o(h) \). The semigroup generated by such a killed process is \( \hat{T}_t f(x) = \Pi_x \alpha_t f(\xi_t) \).

### The \( h \)-transform.
Let \( \xi = (\xi_t, \Pi^h_x) \) be a diffusion killed at \( \tau_E \). Let \( p_t(x, dy) \) be its transition function. Let \( h \in \mathcal{H}(E) = \{ \text{positive harmonic functions in } E \} \). Then let
\[
p^h_t(x, dy) = \frac{1}{h(x)} p_t(x, dy) h(y).
\]

**Exercise 3.3.** Show that
\[
\int_E p_t(x, dy) h(y) \leq h(x),
\]
and therefore \( p^h_t(x, E) \leq 1 \).

The corresponding Markov process, \((\xi_t, \hat{\Pi}^h_x)\) is given by
\[
\hat{\Pi}^h_x \{ \xi_t \in B_1, \ldots, \xi_{t_n} \in B_n \} = \frac{1}{h(x)} \int_{B_1 \times \cdots \times B_n} p_{t_1}(x, dz_1)p_{t_2-t_1}(z_1, dz_2) \cdots p_{t_n-t_{n-1}}(z_{n-1}, dz_n) h(z_n)
\]
for all \( 0 < t_1 < t_2 < \cdots < t_n \) and all Borel \( B_1, \ldots, B_n \subset E \). Recall that \( h \) is a strictly positive harmonic function. Nevertheless, if we wish to extend to \( h \) which may vanish, and therefore define \( \Pi^h_x = h(x)\hat{\Pi}^h_x \). Thus, \( \Pi^h_x(\Omega) = h(x) \).

**Properties.**

(1)
\[
\hat{\Pi}^h_x 1_{t < \tau_E} Y = \frac{1}{h(x)} \Pi_x 1_{t < \tau_E} Y h(\xi_t),
\]
for every \( Y \in \mathcal{F}_t \) (generated by \( \xi_s, s \leq t \)).

(2) For every stopping time \( \sigma \), and every \( \sigma \)-positive \( Y \),
\[
\Pi^h_x 1_{\sigma < \tau_E} Y = \Pi_x 1_{\sigma < \tau_E} Y h(\xi_\sigma).
\]

Property 2 implies (via Feller) that \((\xi_t, \hat{\Pi}^h_x)\) is a continuous strong Markov process. The proofs can be found in [D1], Chapter 7, Lemma 3.1.

### Conditional Diffusion.
Fix a smooth \((C^2)\) domain \( E \), and put \( \xi_{\tau_E} = \eta \). Let \( k_E(x, z) = k_z(x) \) be the Poisson kernel \((x \in E, z \in \partial E) \). So \( k_z \) is harmonic in \( E \). Recall that
\[
\Pi_x \varphi(\eta) = \int_{\partial E} k_z(x) \varphi(z) S(dz),
\]
where \( S \) is normalized surface measure. This implies that
\[
\Pi_x k_\eta(y) \varphi(y) = \int_{\partial E} k_z(x) k_z(y) \varphi(z) S(dz),
\]
which is symmetric in \( x \) and \( y \).

**Theorem 3.4.** For every \( Y \in \mathcal{F} \), and \( f \in \mathcal{B}(\partial E) \),
\[
\Pi_x Y f(\eta) = \Pi_x (f(\eta) \hat{\Pi}^h_x Y).
\]
Here, $\Pi^x_z$ is the $k_z$-transform $\Pi^k_{x_z}$. Hence, $\hat{\Pi}^y_x Y = \Pi_x \{ Y^y \}$, $\Pi_x$-a.s. Hence, $(\xi_t, \hat{\Pi}^y_x)$ can be interpreted as the diffusion started from $x$ and conditioned to exit $E$ at $z$.

**Proof.** By the multiplicative systems theorem, it is sufficient to prove the formula for $Y \in Q$, where $Q$ is a multiplicative system generating $\mathcal{F}$. In fact, $Q = \{ Y' \mathbb{1}_{E > t} ; Y' \in \mathcal{F}, t > 0 \}$ is such a family. Applying property 1 to $h = k_z$, we then have

$$\hat{\Pi}^y_x Y = \frac{1}{k_z(x)} \Pi_x Y k_z(\xi_t),$$

which implies that

$$\hat{\Pi}^{\eta(\omega)}_x Y = \frac{1}{k_{\eta(\omega)}(x)} \int_{\Omega} \Pi_x (d\omega') Y(\omega') k_{\eta(\omega)}(\xi_t(\omega')).$$

By Fubini’s theorem, it follows that

$$\Pi_x f(\eta) \hat{\Pi}^y_x Y = \int \Pi_x (d\omega') Y(\omega') \int \Pi_x (d\omega) \varphi(\eta(\omega)) k_{\eta(\omega)}(\xi_t(\omega')),$$

where $\varphi(z) = f(z)/k_z(y)$ with $y = \xi_t(\omega')$. The inner integral is equal to

$$\Pi_x f(\eta) \hat{\Pi}^{\eta(y)}_x Y = \Pi_y \int_{\partial E} f(z) k_z(\xi_t) S(dz) = \Pi_y \Pi_\xi f(\eta) = \Pi_y f(\eta).$$

4. **Superdiffusions and Nonlinear PDE**

4.1. **BEM systems and superprocesses.**

4.1.1. **Exit systems associated with branching particle systems.** (copy from his notes)

4.1.2. **Branching Exit Markov (BEM) systems.** (The Branching property can be deduced from an appropriate continuity condition for the dependence of $P_\mu$ on $\mu$, but this condition is not very easy to state, let alone prove.)

**Lemma 4.1.** If $w(x) = \Pi_x [e^{-kT} u(\xi_T) + \int_0^T e^{-ks} v(\xi_s) \, ds]$, then

$$w(x) + \Pi_x \int_0^T k w(\xi_s) \, ds = \Pi_x \left[ u(\xi_T) + \int_0^T v(\xi_s) \, ds \right].$$

**Proof.** Let $Y = e^{-kT} u(\xi_T) + \int_0^T e^{-ks} v(\xi_s) \, ds$, so that $w(x) = \Pi_y Y$. Then, using the Markov property,

$$\Pi_x \int_0^T k w(\xi_s) \, ds = k \int_0^\infty \Pi_x (\mathbb{1}_{s < \tau} \Pi_\xi Y) \, ds = k \int_0^\infty \Pi_x \mathbb{1}_{s < \tau} \theta_s Y \, ds.$$

Now, $\{ s < \tau \} \subset \{ \theta_s \tau = \tau - s, \theta_s \xi_T = \xi_T \}$, and this in turn is contained in the set of $s$ such that

$$\theta_s Y = e^{-k(\tau-s)} u(\xi_T) + \int_0^\infty \mathbb{1}_{t < \tau-s} e^{-kt} v(\xi_{t+s}) \, ds = e^{-k(\tau-s)} u(\xi_T) + \int_s^\tau e^{-k(\tau-s)} v(\xi_t) \, dt.$$
Substituting this expression for \( \theta_s Y \),
\[
\Pi_x \int_0^\tau kw(\xi_s) \, ds = \Pi_x \left( \int_0^\tau ke^{-k(\tau-s)} \, ds \right) u(\xi_\tau) + \int_0^\tau ke^{-k(\tau-s)} v(\xi_s) \, ds \, dr
\]
\[
= \Pi_x \left[ (1 - e^{-k\tau}) u(\xi_\tau) + \int_0^\tau (1 - e^{-kr} v(\xi_r) \, dr \right]
\]
\[
= \Pi_x \left[ u(\xi_\tau) + \int_0^\tau v(\xi_r) \, dr - w(x) \right].
\]

This, together with the fundamental equation, implies that
\[
e^{-V_D f(x)} = \Pi_x \left[ k \int_0^\tau \phi(e^{-V_D f(\xi_s)}) \, ds + e^{-f(\xi_\tau)} \right],
\]
where \( \phi(z) = \varphi(z) - z \).

Now, replace each founding particle by \( \ell \) particles (at the same position), each of mass \( 1/\ell = \beta \). Consider the new system \( X^\beta \) consisting of \( (X^\beta_D, \mathbb{P}^\beta) \), where \( \mathbb{P}^\beta = \mathbb{P}_{\mu/\beta} \) and \( X^\beta = \beta X_D \). Then
\[
\mathbb{P}^\beta_{\mu}(f,X^\beta_D) = \mathbb{P}_{\mu/\beta}(f,\beta X_D) = e^{-\beta(V_D(\beta f),\mu/\beta)} = e^{-V_D^\beta(f,\mu)}.
\]
That is, we get a BEM system \( X^\beta \) with transition operator \( V^\beta_D(f) = \frac{1}{\beta} V_D(\beta f) \). (The Markov property must also be checked, but this is straightforward.) Replacing \( f \) with \( \beta f \) in Equation 4.1, we get
\[
e^{-\beta v_D(x)} = \Pi_x \left[ k \int_0^\tau \phi(e^{-\beta v_D(\xi_s)}) \, ds + e^{-\beta f(\xi_\tau)} \right],
\]
where \( v_\beta = V^\beta_D f \). This is equivalent to
\[
u_\beta(x) + \Pi_x \int_0^\tau \psi_\beta(u_\beta(\xi_s)) \, ds = \Pi_x f_\beta(\xi_\tau),
\]
where \( \beta u_\beta = 1 - e^{-\beta v_\beta} \) and \( \beta f_\beta = 1 - e^{-\beta f} \), and
\[
\psi_\beta(u) = \frac{k_\beta}{\beta} [\varphi(1 - \beta u) - 1 + \beta u]
\]
for \( \beta u < 1 \). Of course, this is (by definition) the statement \( u_\beta + G_D \psi_\beta(u_\beta) = K_D f_\beta \).

Note, we expect the parameters \( \varphi \) and \( k \) to depend on \( \beta \). Since \( \beta u_\beta = 1 - e^{-\beta v_\beta} \leq 1 \), the value \( \varphi_\beta(1 - \beta u_\beta) \) is defined. Now, let \( \beta \to 0 \). Then \( f_\beta \to f \). If \( \psi_\beta \) converges to some \( \psi \), then we expect that \( u_\beta \) tends to a limit \( u \) which is a solution of the equation
\[
u + G_D \psi(u) = K_D f.
\]

4.2. A class of superprocesses.

**Theorem 4.2.** A \((\xi,\psi)\)-superprocess exists for every function
\[
\psi(u) = bu^2 + \int_0^\infty (e^{-\lambda u} - 1 + \lambda u) \, d\nu(\lambda),
\]
where \( b \geq 0 \) and \( \nu \) is a measure satisfying
\[
\int_0^\infty \lambda \land \lambda^2 \, d\nu(\lambda) < \infty.
\]
Note, the family of functions \( \psi \) satisfying the conditions of the theorem includes all the functions \( \psi(u) = \text{const.} u^\alpha \) for \( 1 < \alpha < 2 \). Thus corresponds to the choice \( b = 0 \) and \( d\nu(\lambda) = \lambda^{-(1+\alpha)} \, d\lambda \).

fill in from lecture 14 notes

5. Superdiffusions and the Equation \( Lu = \psi(u) \)

fill in from lecture 14 notes

5.1. Time-space superdiffusion.

5.1.1. Time-space process. Let \( \xi = (\xi_t, \Pi_x) \) be a Markov process on \( E \). Denote by \( \eta \) the pair \((t, \xi_t)\) in \( S = \mathbb{R} \times E \). The corresponding probability measures are now doubly indexed, \( \Pi_{r,x} \) in \( \Omega \). Let \( \mathcal{F}_{\leq t} = \sigma\{\eta_s; s \leq t\} = \sigma\{\xi_s; s \leq t\} \) and \( \mathcal{F}_{\geq t} = \sigma\{\eta_s; s \geq t\} = \sigma\{\xi_s; s \geq t\} \). The Markov property for such a process may then be states as

\[
\Pi_{r,x}(Y | Z) = \Pi_{r,x} [Y | \Pi_{t,\xi_t} | Z], \quad \text{for } Y \in \mathcal{F}_{\leq t}, \, Z \in \mathcal{F}_{\geq t}.
\]

If \( Q \) is open in \( S \), then the first exit time from \( Q \) is defined as \( \inf \{ t; \eta \notin Q \} \).

5.1.2. Time-space superprocess. Let \( Q \) be open in \( S \), and let \((X_Q, P_\mu)\) be a \( \psi \)-superprocess (where \( \mu \) are finite Borel measures on \( S \)). Then

\[
P_\mu e^{-\langle f,Q \rangle} = e^{-\langle V_Q f, \mu \rangle},
\]

where \( V_Q(r,x) = -\log P_{r,x} e^{-\langle f,X_Q \rangle} \). Then \( u = V_Q \) satisfies the integral equation

\[
u + G_Q \psi(u) = K_Q f,
\]

where \( G_Q f(r,x) = \Pi_{r,x} \int_0^1 f(s, \xi_s) \, ds \), and \( K_Q f(r,x) = \Pi_{r,x} f(\tau, \xi_\tau) \) (\( \tau = \tau_Q \)). Then, as we have essentially shown, Equation 5.1 implies that \( u \) satisfies the differential equation

\[
\frac{\partial u}{\partial t} + Lu = \psi(u).
\]

5.2. On the integral equation 5.1. Define the set \( \mathcal{Q}_0 = \{ Q; Q \subseteq (r_1, r_2) \times E \} \), for some fixed \( r_1 < r_2 \).

**Theorem 5.1.** Suppose \( \psi(0) = 0 \), \( \psi \) is monotone increasing, and \( |\psi(t_1) - \psi(t_2)| \leq q(c)|t_1 - t_2| \) for \( t_1, t_2 \in [0,c] \). Then Equation 5.1 has a solution for every \( f \in B_c \equiv \{ f; 0 \leq f \leq c \} \) and \( Q \subset [0,c] \times E \).

**Proof (by monotone iteration scheme).** Let \( q(c) = k \). Put

\[
T(u) = \Pi_{r,x} \left[ e^{-k(\tau-r)} f(\tau, \xi_\tau) + \int_r^\tau e^{-k(s-r)} \varphi(u(s, \xi_s)) \, ds \right],
\]

where \( \varphi(u) = ku - \psi(u) \) on \([0,c]\). Consider \( u_0 = 0 \), and \( u_n = T(u_{n-1}) \) for \( n \geq 1 \). It can then be proved (using the Lipschitz condition) that \( 0 = u_0 \leq u_1 \leq \cdots \leq u_n \leq \cdots \leq c \). It follows that \( u_n \uparrow u \) for some \( u \) which satisfies \( u = Tu \). This implies the integral equation 5.1. For details, see [D1], pg. 50.

We would now like to address the uniqueness of the solution in the above theorem. To that end, consider the following lemma.
Lemma 5.2 (Gronwall). Suppose $Q \in \mathbb{O}_0$, and $h \in B_c$. If, for some constants $p, q$, it is true that

$$h(r,x) \leq p + q \Pi_{r,x} \int_r^\tau h(s,\xi_s) \, ds \quad \text{for all } (r,x) \in Q,$$

then

$$h(r,x) \leq p \Pi_{r,x} e^{q(\tau-r)}.$$

Proof (for $p = 0$).

6. The Mean Value Property

Theorem 6.1. If $Lu = \psi(u)$ in $E$, then $V_D u = u$ for every $D$ compactly contained in $E$.

In the proof of this theorem, we will use the following results on parabolic equations. Let $Q = (0,t) \times D$, $w = G_Q \rho$, $h = K_Q f$, and $\partial r Q = \partial Q \setminus \{0\} \times D$.

1. If $\rho \in C^1(Q)$, then $\check{w} + Lw = -\rho$ in $Q$.
2. If $f$ is bounded, then $\check{h} + Lh = 0$ in $Q$.
3. If $D$ is regular, then $w = 0$ and $h = f$ on $\partial r Q$.
4. If $\check{u} + Lu = 0$ in $Q$ and $u = 0$ on $\partial r Q$, then $u = 0$ in $Q$.

Lemma 6.2. Under the conditions of Theorem 6.1, $u + G_Q \psi(u) = K_Q(u)$ for $Q = (0,t) \times D$.

Proof. Suppose $D$ is regular. Put $F = u + G_Q \psi(u) - K_Q(u)$. By items 1 and 2 above, $\check{F} + LF = 0$ in $Q$. By item 3, it follows that $F = 0$ on $\partial r Q$, and hence by item 4, $F = 0$ in $Q$. In the more general case (where $D$ is not necessarily regular), we consider a sequence of regular domains $D_n$ exhausting $D$, and we pass to a (monotone) limit in $u + G_{Q_n} u = K_{Q_n} u$.

Proof of Theorem 6.1.

References


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