2. Projective geometry and the extended Euclidean plane

As we can see from Hilbert's treatment, a completely worked out axiom system for geometry in the plane is quite complicated. By the nineteenth century it was realized that even the first few axioms of Euclid's treatment led to interesting results, and the problems that came up in properly drawing a picture on a flat canvas led to a somewhat different way of presenting the axiom system. In particular, the axioms of "incidence" can be stated in such a way that they are very simple and yet lead to interesting non-trivial results

By definition, parallel lines never meet. On the other hand, when one looks at or draws parallel lines, they do seem to meet. Train tracks seem to meet on the horizon. The horizon itself seems to limit the extent of the plane. So why not incorporate these feelings into the language of Geometry? Euclid's first axiom states that any two distinct points lie on a unique line. In other words they are incident to a unique line. But do two distinct lines determine a unique point? Usually, but not always. Most pairs of distinct lines intersect in a unique point. They are incident to that point of intersection. But of course, no point in sight is incident to distinct parallel lines.

Why not create new points that can be incident to parallel lines, such as the points on the horizon seem to be? In fact, we can do just that without giving up any mathematical precision, although some of the statements using this language sound a bit strange when you first hear them.

In Euclidean geometry, all lines parallel to a fixed line are parallel to each other. So the collection of all lines in the plane fall into equivalence classes, determined by the property of being parallel. (For our convenience we say that a line is parallel to itself.) In other words, we have:

1. If L_1 and L_2 are two lines and L_1 is parallel to L_2, then L_2 is parallel to L_1.

2. If L is a line, then we can say that L is parallel to L itself.

3. Suppose L_1, L_2, and L_3 are lines. If L_1 is parallel to L_2, and L_2 is parallel to L_3, then L_1 is parallel to L_3.

In the language of set theory, the relation "is parallel to" is an equivalence relation, since it satisfies the symmetric, reflexive, and transitive properties, 1, 2, 3, respectively. So they determine equivalence classes.

Corresponding to each equivalence class we associate what we call a *point at infinity*, or an *ideal point*. We also say that each point at infinity is incident to every line in its corresponding equivalence class. We also say that the collection of all the points at infinity form a single *line at infinity*. Lastly, we say every point at infinity is incident to this line at infinity.

This almost corresponds to our intuition about points on the horizon. If we stand in the middle of a pair of train tracks, they appear to meet at one point one the horizon in front of us, but if we turn around, they also appear to meet at another point on the horizon. Our definition, however, only allows for one point at infinity for these parallel train tracks. In our definition, these two "points on the horizon" are identified as a single point at infinity. Imagine a train disappearing over the horizon in one direction, only to reappear from the opposite direction.

Summarizing, we now have added infinitely many new points to the Euclidean plane, which we called the points at infinity. We will call the Euclidean points ordinary points to distinguish them from the points at infinity. We have also added one new line, the line at infinity. We call the Euclidean lines ordinary lines to distinguish them from the line at infinity.

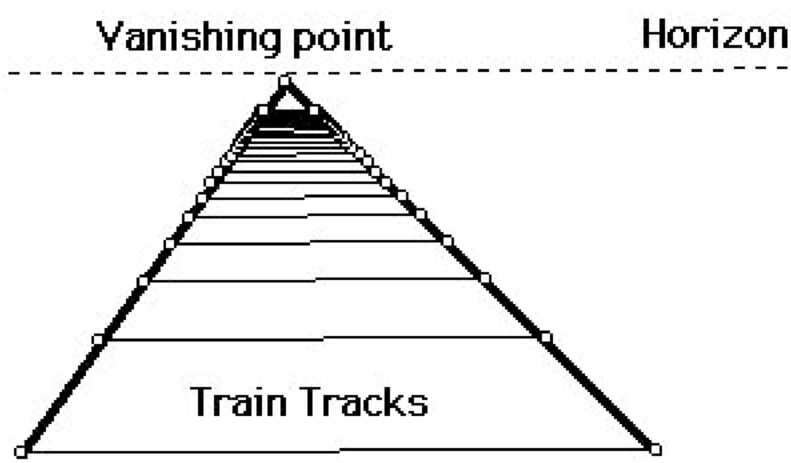

Figure 1:

1 Exercises

1. Show that for every pair of distinct lines, ordinary or not, there is a unique point that is incident to both of them.

2. Show that for every pair of distinct points, ordinary or not, there is a unique line for which they are both incident.

3. Fix a line L in the Euclidean plane. We say that two lines L_1 and L_2 are "equivalent" if either L_1 and L_2 are both incident to the same point on L or they are both parallel to L. Show that this is an equivalence relation. How is this related to the equivalence relation above?