R. Connelly Math 452, Spring 2008

Classical Geometries

1 Greek art and geometry

In Plato's view, only that which turned the soul's eye from the material world to objects of pure thought was worthy of a philosopher's study. Music, astronomy, arithmetic, and especially geometry were particularly recognized. The idea was to start with certain unquestioned assumptions and precise definitions and proceed logically and rigorously from there.

Euclid of Alexandria, a student of the Platonic school, lived from 428 BC to 348 BC. Around 300 BC he wrote the Elements, a collection of thirteen books in which he compiled the geometry and number theory developed in previous centuries and presented it in an axiomatic way. His work has had a profound and lasting effect on almost all of western mathematics. Included in this handout is a copy of the beginning of Book I of the Elements, as well as a later "revised" edition by Playfair. It is ironic that Euclid's first Theorem, about constructing an equilateral triangle, has a hidden assumption which is just what he was apparently trying to avoid.

Mathematicians eventually got around to looking critically at Euclid's work. One of the most detailed and thoughtful of these people was David Hilbert, who is considered among the strongest and surely one of the most influential mathematicians in the nineteenth and twentieth centuries. Included is a copy of the beginning of Hilbert's Foundations of Geometry, which is his attempt to achieve Euclid's goal of presenting geometry as following from a system of coherent axioms, with present day rigor. Hilbert has conveniently separated the axioms into classes of incidence, order, congruence, parallels, and continuity. But despite this heroic attempt at a natural and reasonable presentation, we see that the system is quite complex and involves some quite profound ideas.

Meanwhile, from a different perspective, we have a few pictures of Egyptian Art and Greek Art. Plato referred to Art as the "worthless mistress of a worthless friend, and the parent of a worthless progeny." [Perhaps some architecture students today feel the same about Mathematics (or at least Calculus).] Looking at these few examples of Greek vase painting, possibly Greek painting was at a primitive level, inviting such attacks as Plato's. On the other hand, sculpture was very "realistic" and seems to have been much more advanced. Some of the best "drawings" seem to be obtained by sticking a sculpture on a slab, blurring the distinction between Drawing and Sculpture. Apparently the Greeks, both Mathematicians and Artists, did not have a good grasp of what we today call "perspective."

It is my assertion that an understanding of perspective in one area, Mathematics or Art, could have influenced the other; but this did not happen until well over a thousand years after the prime of Greek Mathematics, during the Renaissance. After the time of Euclid and Archimedes, with a few individual twitches only, Greek Mathematics died. It was preserved and copied with great reverence, but it did not grow or develop until after the great Dark Ages. It took the simple, straightforward curiosity of people such as Flippo Brunellischi, the builder of the large dome in Florence, Leonardo da Vinci, famous for the Mona Lisa, Alberti, who wrote a very influential manual on the principles of perspective drawing, and Dürer to see beyond the strict rules laid down by Euclid and to do what was necessary to draw a picture.

As Euclid's Elements was copied, it was studied intensely. Not only were "mistakes" found, but the question of the necessity of the fifth Postulate arose, and this eventually became a major unsolved problem. Without assuming something in its place, could Euclid's fifth Postulate be proved from the previous four? Many people tried in vain. In the nineteenth century, Gauss showed that the fifth Postulate could be proved if one assumed that there was a triangle of arbitrarily large area. Similarly, Legendre proved the fifth Postulate, but he assumed that there was a triangle whose angles added up to 180 degrees. There were many more attempts. It was not until the nineteenth century that people realized that it may not be possible to prove the fifth Postulate from the others at all. Indeed, the belief in the "absolute truth" of the axioms were gradually replaced with the more formalist idea that meanings of the axioms could be put in different contexts. Eventually it was shown that if the system with the fifth Postulate is consistent, then it is impossible to prove the fifth Postulate from the others. In other words, if one has a consistent system with Euclid's fifth Postulate, then one can prove the existence of a consistent system with an alternative axiom that directly contradicts Euclid's.

The proof (of this impossibility of a proof) is very simple. Playfair's version of the fifth Postulate says that "Given any line and a point not on the line, there is one and only one line through the point parallel to the first line." In the Euclidean plane one constructs a "model" which is a system of points and lines (and whatever else is needed) that satisfies all of Euclid's axioms, except that the fifth is replaced by a property which states that "Given any line and a point not on the line, there is more than one line through the point parallel to the first line." See Figure 1. We will see this in more detail later; this model is usually called the "hyperbolic plane."

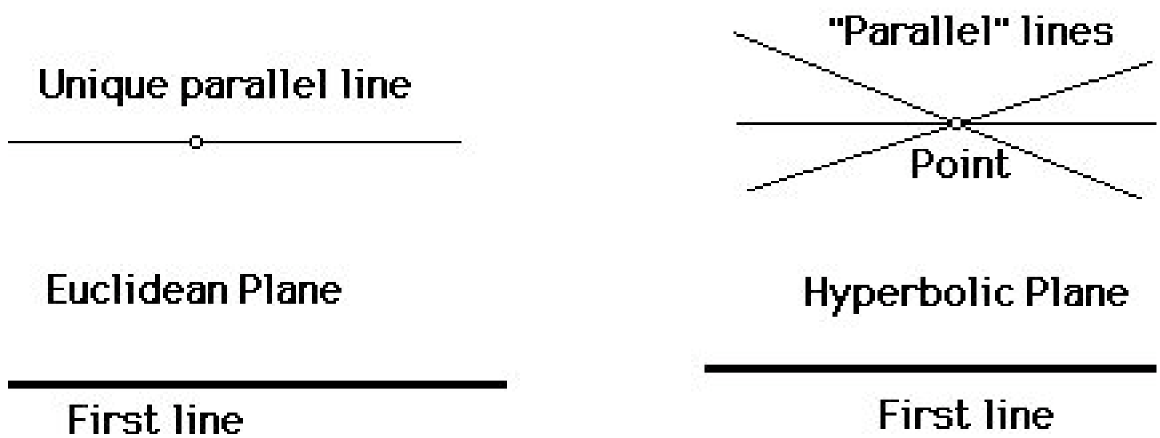

Figure 1: This shows the difference in a geometry where the parallel postulate does not hold, on the right, and where it does hold, on the left.

Closely related to the hyperbolic plane is what is now called the "projective plane" or the "elliptic plane." It too can be regarded as a model to show that Euclid's fifth Postulate does not follow the others, if one has a very liberal interpretation of the axioms (which is almost never done). Here the fifth Postulate is replaced by the statement "Any two distinct lines intersect in a unique point." The model of a "line" is taken to mean a great circle (the intersection with the sphere of a plane through its center) on a sphere, and a "point" is taken to mean a pair of opposite points on the sphere. The notions of "projection" and a convenient treatment of the ideas developed by the Renaissance artists are most naturally at home in the projective plane.

1.1 Exercises:

1. Explain what is wrong with the projective plane as a model satisfying the first four Euclidean Postulates.
2. How would you prove Euclid's first Theorem, if you were Euclid and had a second chance.
3. Discuss the apparent distortions in the photocopies of the Egyptian and Greek paintings. Is there a simple way to correct the "mistakes"?
4. Draw a "square" in a horizontal plane as it is seen in a vertical plane.
5. Draw a "cube" as accurately as you can.
6. Draw a truncated pyramid with a triangular base.