Groups with f.p. properties

Janusz Kiel FAQ

G. Arzhantsev, M. Bridson, T. J. Leary, A. Minasyan, K. Sviridov

T. J. - P. Kropholler, T. L.

Serre's exercise.

(W, S) a Coxeter gp. When does it have a f.p. free action on a tree?

Relations: $s_i^2 = 1$, $(s_i s_j)^{m_{ij}} = 1$. If one $m_{ij}$ is $\infty$, then $W$ splits

$$W = W_{S-\{i\}} \ast W_{S-\{i, j\}}$$

What happens when all $m_{ij}$ finite?

Then any action on $T$ has a fixed point.

- $W_{S-\{i\}}$ has nonempty Fix $\{s_j\}$ ($v \mapsto s_j v \mapsto v$ so midpt of geodesic is fixed)

- Fix $\{s_j\} \cap$ Fix $\{t_j\} = \text{Fix} \{W_{S-\{i, j\}}\}$

- Helly implies that $\bigcap_{\text{set } S} \text{Fix}(S) \neq \emptyset$.

- Every 3 gens shape a f.p.

Suppose not: Wiser! has empty f.p. set.

Fix are convex sets; pairwise intersection.

$H^1(U \text{Fix}_S) \neq 0$,

contradiction.
Generalize this? Action of Coxeter groups on other spaces? Need convexity - how about CAT(0)?

Farb, Barnhill (Bridson):

\[(W,S) \rightleftharpoons W(S_1, \ldots, S_k)\] finite groups.

then any isometric action on CAT(0) space of dim. \(\frac{k-1}{2}\) has a fp.

Pf similar to tree. intersection of convex sets give nontrivial homology in dim k, bad.

**Ex** \(\widetilde{A_n} \rightleftharpoons \text{rel gen } \mathbb{A}_n \subset \mathbb{R}^n\).

Any isometric action on \(X \rightleftharpoons \text{CAT(0)}\) space has a fp.

(In T.S., P.K.-L.): If \(Q\) is a common quotient of all \(\mathbb{A}_n\), then \(Q\) has a fp for any isometric action on any fin dim CAT(0) space.

Common quotients trivial. \(\mathbb{Z}_2\). No others?

[AM 03, 05]: Given \(E \cong G_0 \cap \cdots \cap G_n\) nonelementary, hyp.; \(H\), any countable infinite gp. Then \(\exists Q\) s.t. \(G_n \rightarrow Q \leftarrow H\). \(\forall n\)

\(Q\) simple, Kazhdan \(\overline{(Q, \text{periodic})}\)

\(\exists\) Lefschetz

Every \(G_n \rightarrow Q \leftarrow C\rightarrow H\), \(C_n\) rel \(H\) hyp.
There is a seq on $G_n$ non-elm., hyp, st $G_n$ is gen by $n+1$ elts, 
every $n$-elt subset spans a finite gp.
(Helly argument) thus $G_n$ has a fp on any $CA^2(0)$ space of dim $< n$.

Smith Theory $\xrightarrow{\text{p-acyclic}}$ fin dim, $G$ finite p-gp.

Then $\tilde{H}^k(X, \mathbb{Z}/p\mathbb{Z}) = 0$.

$\emptyset \neq X$, $X$ is p-acyclic.

(has more to it, but this is all we need)

(1) Suppose $\{G_n\}_n$ is non-elm. hyp gen by $(n+1)$-elts, and every 
k-elt set $k < n+1$ spans a finite p-gp. Then

$G_n$ acting on $X^{n+1}$ p-acyclic, dim $X < n$ has a fp.

hence any common quotient $Q$ of $G_n$ has a fp

for any action on a fin dim $p$-space

Ex. $Q$, $GQ$ by translations are effective.

Remark If $Q$ (from (1)) acts simplicially on a locally finite complex,

then the action is trivial.

Pf $Q$ has a fp. Simplicial $\Rightarrow$ stabilizers combine to form

balls $B(r, x)$, hence is an identity.

If $Q$ acts on a contractible fin dim $p$-mfd, then the action
is trivial (Pf as above - use $\{x\}$ formal power series)
There are \( G_n \) non-edge hyp. gen by \((x, 1)\), \( e \), every knot set kernel spans finite \( p \)-gp, \( G_n \) acting on \( X^{n-1} \).

Developable simplices of \( G \) 

\[
S \text{ a set, } T \rightarrow G_T, \quad G_T \text{ gen by } T
\]

\[
T : T' \rightarrow G_T \hookrightarrow G_{T'}, \quad \text{to simplify these inclusions, gens } s \rightarrow \text{ gens.}
\]

"inductively constructing bigger gps."

Given \( G_T \), many choices for \( G_S \). But there is universal "biggest" one 

\[
G_T \hookrightarrow G_S, \quad G_S \rlap{.}
\]

Ex Coxeter gps.

The space associated with \( G_n \) is \( \Delta^S \times G_S / \sim \)

\[
(p, g) \sim (g', h) \iff p = q, \quad g' h \in G(p).
\]

Retractable simplex of \( G \) 

\[
r_T(t) = \begin{cases} t & \text{if } t \in T \\ G_S & \text{otherwise.} \end{cases}
\]

\( r_T \) extends to a homomorphism \( r_T : G_S \rightarrow G_T \).

Loc Retractable \( \Rightarrow \) developable \( \Rightarrow G_S \) retractable.

"Doesn't correctly..." \( G_S \) is direct prod. 

\[
G_T \Rightarrow G_S
\]

\[
\uparrow \quad G_S
\]

"Do it correctly..." 

\[
\text{If } G_T \text{ are finite, so is } G_S
\]

\[
\text{If } G_T \text{ are } p \text{-gps, so is } G_S.
\]