On asymptotic dimension of Coxeter groups

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Outline of the talk

- Main result
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- Asymptotic dimension
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- Estimate of $\text{asdim}$ from below
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- Asymptotic dimension
- Estimate of \( \text{asdim} \) from below
- Estimate of \( \text{asdim} \) from above
- Amalgamation theorem
Main Result

AMALGAMATION THEOREM.

\[ \text{asdim}(A \ast_C B) \leq \max\{\text{asdim } A, \text{asdim } B, \text{asdim } C + 1\}. \]
Main Result

- **AMALGAMATION THEOREM.**
  
  \[ \text{asdim}(A *_C B) \leq \max\{\text{asdim } A, \text{asdim } B, \text{asdim } C + 1\}. \]

- **THEOREM.** *For Coxeter groups* \((\Gamma, S)\),

  \[ \text{asdim } \Gamma \leq \dim \Sigma(\Gamma, S) = \dim N(\Gamma, S) + 1 \]

  *where* \(\Sigma(\Gamma, S)\) *is the Davis complex and* \(N(\Gamma, S)\) *is the nerve.*
Coxeter groups

A Coxeter matrix \((m_{uv})_{u,v \in S}\) is a symmetric matrix with 
\(m_{vv} = 1\) and \(m_{uv} \in \mathbb{N} \cup \{0\}\). A Coxeter matrix is 
right-angled if \(m_{uv} \in \{0, 2\}\) for \(u \neq v\).
A Coxeter matrix \((m_{uv})_{u,v \in S}\) is a symmetric matrix with \(m_{vv} = 1\) and \(m_{uv} \in \mathbb{N} \cup \{0\}\). A Coxeter matrix is right-angled if \(m_{uv} \in \{0, 2\}\) for \(u \neq v\).

A Coxeter matrix defines a Coxeter group

\[
\Gamma = \langle S \mid (uv)^{m_{uv}}, u, v \in S \rangle
\]

Thus \(s^2 = 1 \forall s \in S\).

\(\Gamma\) is generated by the set of reflections \(S\).
Nerve $N = N(\Gamma, S)$ of a Coxeter group $\Gamma$ is a finite complex with the set of vertices $S$. Vertices $u_1, \ldots, u_k$ span a simplex iff they generate a finite subgroup in $\Gamma$. Thus, $u$ and $v$ are joined by an edge iff $m_{uv} \neq 0$. In right-angled case fill all empty triangles, then 3-simplices, then 4-simplices and so on.
Nerve $N = N(\Gamma, S)$ of a Coxeter group $\Gamma$ is a finite complex with the set of vertices $S$. Vertices $u_1, \ldots, u_k$ span a simplex iff they generate a finite subgroup in $\Gamma$. Thus, $u$ and $v$ are joined by an edge iff $m_{uv} \neq 0$. In right-angled case fill all empty triangles, then 3-simplices, then 4-simplices and so on.

The Davis complex $X$ is the image of a simplicial map $q : \Gamma \times Cone(\beta N) \to X$ define by the equivalence relation on vertices: $\gamma \times c_\sigma \sim \beta \times c_{\sigma'}$ iff $\sigma = \sigma'$ and $\gamma^{-1}\beta \in \Gamma_\sigma$ where $\sigma, \sigma' \subset N$ are simplices and $c_\sigma, c_{\sigma'}$ are their barycenters. $\Gamma_\sigma \subset \Gamma$ is generated by $\sigma$. 
Nerve $N = N(\Gamma, S)$ of a Coxeter group $\Gamma$ is a finite complex with the set of vertices $S$. Vertices $u_1, \ldots, u_k$ span a simplex iff they generate a finite subgroup in $\Gamma$. Thus, $u$ and $v$ are joined by an edge iff $m_{uv} \neq 0$. In right-angled case fill all empty triangles, then 3-simplices, then 4-simplices and so on.

The Davis complex $X$ is the image of a simplicial map $q : \Gamma \times Cone(\beta N) \rightarrow X$ define by the equivalence relation on vertices: $\gamma \times c_\sigma \sim \beta \times c_{\sigma'}$ iff $\sigma = \sigma'$ and $\gamma^{-1}\beta \in \Gamma_\sigma$ where $\sigma, \sigma' \subset N$ are simplices and $c_\sigma, c_{\sigma'}$ are their barycenters. $\Gamma_\sigma \subset \Gamma$ is generated by $\sigma$.

A Coxeter group acts on its Davis complex by reflections.
Example
For every Coxeter group \( \text{asdim} \Gamma < \infty \).
asdim of Coxeter groups

- (Dr. - Januszkiewicz) *For every Coxeter group* $\text{asdim} \Gamma < \infty$.

- What is $\text{asdim} \Gamma$ of Coxeter groups?
Asymptotic dimension

**DEFINITION. (Gromov)** \( \text{asdim} X \leq n \) if for every uniformly bounded cover \( \mathcal{V} \) of \( X \) there exists a uniformly bounded cover \( \mathcal{U} \) of order \( \text{ord} \mathcal{U} \leq n + 1 \) such that \( \mathcal{V} \prec \mathcal{U} \).

Here \( \text{ord}_x \mathcal{U} = |\{U \in \mathcal{U} \mid x \in U\}| \), and \( \text{ord} \mathcal{U} = \max_{x \in X} \text{ord}_x \mathcal{U} \).

\[
\mathcal{V} \prec \mathcal{U} \iff \forall V \in \mathcal{V} \exists U \in \mathcal{U} : V \subset U
\]
Asymptotic dimension

DEFINITION. (Gromov) \( \text{asdim} X \leq n \) if for every uniformly bounded cover \( \mathcal{V} \) of \( X \) there exists a uniformly bounded cover \( \mathcal{U} \) of order \( \text{ord}_U \leq n + 1 \) such that \( \mathcal{V} \prec \mathcal{U} \).

Here \( \text{ord}_x U = |\{ U \in \mathcal{U} \mid x \in U \}| \), and \( \text{ord} U = \max_{x \in X} \text{ord}_x U \).

\[ \mathcal{V} \prec \mathcal{U} \iff \forall \mathcal{V} \in \mathcal{V} \exists \mathcal{U} \in \mathcal{U} : V \subset U \]

DEFINITION. (Lebesgue) \( \text{dim} X \leq n \) if for every open cover \( \mathcal{V} \) of \( X \) there exists an open cover \( \mathcal{U} \) of order \( \leq n + 1 \) such that \( \mathcal{U} \prec \mathcal{V} \).
DEFINITION. (Gromov) \( \text{asdim} X \leq n \) if for every \( d < \infty \) there exists a uniformly bounded cover \( \mathcal{U} \) such that 
\[ \mathcal{U} = \mathcal{U}_0 \cup \cdots \cup \mathcal{U}_n \]
and each family \( \mathcal{U}_i \) is \( d \)-disjoint.

A family \( \mathcal{A} \) of sets in a metric space \( X \) is \( d \)-disjoint if 
\( \text{dist}(A, B) \geq d \) for all \( A, B \in \mathcal{A}, A \neq B \) where 
\[ \text{dist}(A, B) = \inf \{ \text{dist}(a, b) \mid a \in A, b \in B \} \].
**Equivalent definition**

- **DEFINITION. (Gromov)** \( a\text{sdim} X \leq n \) if for every \( d < \infty \) there exists a uniformly bounded cover \( \mathcal{U} \) such that \( \mathcal{U} = \mathcal{U}^0 \cup \cdots \cup \mathcal{U}^n \) and each family \( \mathcal{U}^i \) is \( d \)-disjoint.

A family \( \mathcal{A} \) of sets in a metric space \( X \) is \( d \)-disjoint if \( \text{dist}(A, B) \geq d \) for all \( A, B \in \mathcal{A}, A \neq B \) where \( \text{dist}(A, B) = \inf \{ \text{dist}(a, b) | a \in A, b \in B \} \).

- **DEFINITION. (Ostrand)** \( \text{dim} X \leq n \) if for every \( \epsilon \) there exists an \( \epsilon \)-small open cover \( \mathcal{U} \) such that \( \mathcal{U} = \mathcal{U}^0 \cup \cdots \cup \mathcal{U}^n \) and each family \( \mathcal{U}^i \) is disjoint.
Coarse invariance

PROPOSITION. $asdim(X) = asdim(Y)$ for coarsely equivalent metric spaces $X$ and $Y$. 
Coarse invariance

**PROPOSITION.** \( \text{asdim}(X) = \text{asdim}(Y) \) for coarsely equivalent metric spaces \( X \) and \( Y \).

Since all word metrics \( d_S \) for finite \( S \) in a f.g. group \( \Gamma \) are quasi-isometric, we obtain that \( \text{asdim}(\Gamma, d_S) \) for finitely generated group \( \Gamma \) does not depend on choice of the finite generating set \( S \).
Asymptotic dimension

THEOREM. [Gromov] \( \text{asdim } X \leq n \) iff \( \forall \epsilon > 0 \) there is a uniformly cobounded \( \epsilon \)-Lipschitz map \( f : X \to K \) to a uniform \( n \)-dimensional simplicial complex. Here

- uniform complex: \( K \subset l_2(K^{(0)}) \);
- uniformly cobounded map \( f : X \to K \): \( \exists C > 0 \) such that \( \text{diam}(f^{-1}(\Delta)) < C \) for all simplices \( \Delta \subset K \).
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THEOREM. [Alexandroff] \( \text{dim } X \leq n \) iff \( \forall \epsilon > 0 \) there is an \( \epsilon \)-map \( f : X \to K \) to an \( n \)-dimensional simplicial complex.

\( \epsilon \)-map \( f : X \to K \): \( \text{diam}(f^{-1}(\Delta)) < \epsilon \) for all simplices \( \Delta \subset K \).
THEOREM. A compact metric space $X$ has $\dim X \leq n$ iff it admits a Čech approximation by $n$-dimensional polyhedra:

$$N_1 \leftarrow N_2 \leftarrow N_3 \leftarrow \ldots.$$
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$$N_1 \leftarrow N_2 \leftarrow N_3 \leftarrow \ldots.$$ 

$N_i$ are the nerves of a system of open covers with $\mathcal{U}_{i+1} \prec \mathcal{U}_i$ and with $\text{diam} \mathcal{U}_i \to 0$. 
THEOREM. \(\text{asdim } X \leq n\) iff \(X\) admits an anti-Čech approximation by \(n\)-dimensional polyhedra:

\[N_1 \to N_2 \to N_3 \to \ldots.\]
THEOREM. $\text{asdim } X \leq n$ iff $X$ admits an anti-Čech approximation by $n$-dimensional polyhedra:

$$N_1 \to N_2 \to N_3 \to \ldots.$$ 

$N_i$ are the nerves of a system of open covers with $U_i \prec U_{i+1}$ and with $L(U_i) \to 0$ where $L(U)$ is the Lebesgue number of $U$.

$$L(U) = \inf_{x \in X} \sup_{U \in U} d(x, X \setminus U).$$
Coarse cohomology

Coarse homology $H X_*:$

$$H X_n(Y) = \lim_{\rightarrow} H_{n}^{lf}(N_i).$$
Coarse cohomology

- Coarse homology $H X_* :$

$$H X_n(Y) = \lim_{\to} H_{n}^{lf}(N_i).$$

- Coarse cohomology

$$0 \to \lim^1 H_{c}^{n-1}(N_i) \to H X^{n}(Y) \to \lim H_{c}^{n}(N_i) \to 0.$$
Coarse cohomology

- Coarse homology $H_{X*}$:

$$HX_n(Y) = \lim_{\to} H^f_{n}(N_i).$$

- Coarse cohomology

$$0 \to \lim_{\leftarrow} H_{c}^{n-1}(N_i) \to HX^n(Y) \to \lim_{\leftarrow} H_{c}^{n}(N_i) \to 0.$$  

- Roe’s THEOREM. $HX^n(Y) = H_c(Y)$ for uniformly contractible $Y$.  
  $Y$ is uniformly contractible if $\forall R \exists S$ such that the inclusion $B_R(x) \subset B_S(x)$ is null-homotopic for all $x \in X$. 

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Estimate of $\text{asdim}$ from below

\begin{itemize}
  \item PROPOSITION. $\text{asdim} \Gamma \geq \text{vcd} \Gamma$ for Coxeter groups $\Gamma$.
\end{itemize}
PROPOSITION. \( \text{asdim} \Gamma \geq vcd \Gamma \) for Coxeter groups \( \Gamma \).

Proof. Let \( vcd \Gamma = n \) and let \( \Gamma' \) be a torsion free finite index subgroup with \( cd \Gamma' = n \).

Then \( H^n_c(\Sigma(\Gamma)) = H^n(B \Gamma', \mathbb{Z} \Gamma) \neq 0 \).

By Roe’s Theorem \( HX^n(\Sigma) \neq 0 \).

Since \( H^n_c(\Sigma) = \check{H}^n(\alpha \Sigma) \), the group \( H^n_c(\Sigma) \) is countable.

Since \( \lim^{1} \) cannot be countable (if \( \neq 0 \)),

\( \lim \leftarrow H^n_c(N_i) \neq 0 \).

Thus \( \dim N_i \geq 0 \). Hence \( \text{asdim} \Gamma \geq n \). \( \square \)
vcd of Coxeter groups

\[ vcd \Gamma = \max \{ \gcd(Lk(\sigma, CN)) \} + 1 \]

where \( CN \) is the cone over the nerve \( N(\Gamma) \).
\[ \text{vcd of Coxeter groups} \]

\[ \text{vcd}\Gamma = \max\{\gcd(Lk(\sigma, CN))\} + 1 \]

where \( CN \) is the cone over the nerve \( N(\Gamma) \).

**global cohomological dimension of a space:**
\[ \text{gcd}K = \max\{n \mid H^n(K) \neq 0\} . \]

\( Lk(\sigma, K) \) is the link of a simplex \( \sigma \subset K = \) the union of all simplices \( \sigma' \) such that \( \sigma' \ast \sigma \subset K \).
asdim vs vcd

Is always asdim $\Gamma = vcd(\Gamma)$?
asdim vs vcd

- Is always $\text{asdim } \Gamma = \text{vcd}(\Gamma)$?
- ’Yes’ for hyperbolic groups (Buyalo-Lebedeva & Bestvina-Mess).
  ’Yes’ for polycyclic groups (Bell-Dr.)
asdim vs vcd

- **Is always** $\text{asdim } \Gamma = \text{vcd}(\Gamma)$?
- ’Yes’ for hyperbolic groups (Buyalo-Lebedeva & Bestvina-Mess).
  ’Yes’ for polycyclic groups (Bell-Dr.)
- **Is** $\text{asdim } \Gamma = \text{vcd}(\Gamma)$ where $\Gamma$ is a right-angled Coxeter group with the nerve an acyclic 2-complex?

Note that $\Gamma$ is a candidate for a counterexamples to the Eilenberg-Ganea problem: $\text{vcd}\Gamma = 2, \text{gd}\Gamma = 3$? It’s unclear if $\text{asdim } \Gamma$ should be with vcd.
Estimate of $\text{asdim}$ from above

Dr- Januszkiewicz: $\text{asdim} \Gamma \leq |S|$
Estimate of $\text{asdim}$ from above

- Dr- Januszkiewicz: $\text{asdim}\, \Gamma \leq |S|$
- Dr- Schroeder: $\text{asdim}\, \Gamma \leq \text{ch}(N(\Gamma))$ where $\text{ch}(K)$ is the chromatic number of the complex (graph) $K$.

$\text{ch}(\text{pentagon}) = 3$ and $\text{asdim}\, \Gamma = \dim(\text{pentagon}) + 1 = 2$. 
Estimate of $\text{asdim}$ from above

THEOREM. $\text{asdim} \Gamma \leq \dim N(\Gamma) + 1$ for right-angled Coxeter groups $\Gamma$. 
THEOREM. \( \text{asdim } \Gamma \leq \dim N(\Gamma) + 1 \) for right-angled Coxeter groups \( \Gamma \).

\textit{Proof.} Induction on \( \dim N(\Gamma) \) and induction on \( |N(\Gamma)^{(0)}| \).

If \( \dim N = 0 \), then \( \Gamma \) is virtually free and \( \text{asdim } \Gamma = 1 \).

Let \( \dim N = n \). Then \( |N^{(0)}| \geq n + 1 \). If \( |N^{(0)}| = n + 1 \), then \( N = \Delta^n \) is the \( n \)-simplex. Then \( \Gamma \) is finite and \( \text{asdim } \Gamma \leq 0 \).

If every vertex \( v \) of \( N \) is connected by an edge with any other vertex, then \( N^{(1)} = \sigma^{(1)} \) for a simplex \( \sigma \) with \( \dim \sigma > n \). Since \( \Gamma \) is right-angled \( N = \sigma \). Contradiction with \( \dim N = n \).
Proof continued. Thus, there is a vertex \( v \in N \) such that \( St(v, N) \) does not contain \( N^{(0)} \). Consider \( N_1 = St(v, N) \), \( N_2 = N \setminus OSt(v, N) \), and \( K = Lk(v, N) \). Then

\[
\Gamma = \Gamma_{N_1} \ast \Gamma_K \Gamma_{N_2}
\]

where \( \Gamma_L \) is a subgroup of \( \Gamma \) generated by \( L^{(0)} \subset S \). Note that \( N(\Gamma_{N_i}) = N_i \) and \( N(\Gamma_K) = K \). By induction \( \text{asdim} \, \Gamma_{N_i} \leq n + 1 \) and \( \text{asdim} \, \Gamma_K \leq n \). By the Amalgamation Theorem,

\[
\text{asdim} \, \Gamma \leq \max \{ \text{asdim} \, \Gamma_{N_i}, \text{asdim} \, \Gamma_K + 1 \} = n + 1.
\]

\[\square\]
WEAK AMALGAMATION THEOREM. [Bell-Dr]

\[ \text{asdim } A \ast_C B \leq \max\{\text{asdim } A, \text{asdim } B\} + 1. \]

The proof is based on the following
WEAK AMALGAMATION THEOREM. [Bell-Dr]

\[ \text{asdim } A \ast_C B \leq \max\{\text{asdim } A, \text{asdim } B\} + 1. \]

The proof is based on the following

ACTION THEOREM. [Bell-Dr] Let \( \Gamma \) act on \( X \) by isometries and let \( \text{asdim } X \leq n \). Suppose that \( \text{asdim}(\text{Stab}_R(x_0)) \leq k \ \forall R > 0 \) where

\[ \text{Stab}_R(x_0) = \{ g \in \Gamma \mid d(g(x_0), x_0) \leq R \} \]

is the \( R \)-stabilizer of \( x_0 \) for some \( x_0 \in X \). Then \( \text{asdim } \Gamma \leq n + k \).
Proof of the Weak Amalgamation Theorem. \( \Gamma = A \ast_C B \) acts on the Bass-Serre tree \( T \) whose vertices are the left cosets \( \Gamma/A \bigsqcup \Gamma/B \) and the vertices \( xA \) and \( xB \), \( x \in \Gamma \), and only them are joined by an edge. Note that \( Stab_R(\{A\}) \subset (AB)^R \), \( R \in \mathbb{N} \). The Action Theorem together with following Lemma complete the proof.
Proof of the Weak Amalgamation Theorem. \( \Gamma = A \ast_C B \) acts on the Bass-Serre tree \( T \) whose vertices are the left cosets \( \Gamma/A \bigsqcup \Gamma/B \) and the vertices \( xA \) and \( xB, \ x \in \Gamma \), and only them are joined by an edge. Note that \( Stab_R(\{A\}) \subset (AB)^R, \ R \in \mathbb{N} \). The Action Theorem together with following Lemma complete the proof.

**Lemma:** \( \text{asdim}(AB)^m \leq \max\{\text{asdim} A, \text{asdim} B\} \).
Weak Amalgamation Theorem

**Proof of the Weak Amalgamation Theorem.** \( \Gamma = A \ast_C B \) acts on the Bass-Serre tree \( T \) whose vertices are the left cosets \( \Gamma / A \amalg \Gamma / B \) and the vertices \( xA \) and \( xB \), \( x \in \Gamma \), and only them are joined by an edge. Note that \( \text{Stab}_R(\{A\}) \subset (AB)^R \), \( R \in \mathbb{N} \). The Action Theorem together with following Lemma complete the proof.

**LEMMA:** \( \text{asdim}(AB)^m \leq \max\{\text{asdim } A, \text{asdim } B\} \).
The proof of the Lemma is based on the following:

UNION THEOREM. Let $X = \bigcup_{\alpha} X_{\alpha}$ be a metric space where the family $\{X_{\alpha}\}$ satisfies the inequality $\text{asdim } X_{\alpha} \leq n$ uniformly. Suppose further that for every $r$ there is a $Y_r \subset X$ with $\text{asdim } Y_r \leq n$ so that $d(X_{\alpha} \setminus Y_r, X_{\alpha'} \setminus Y_r) \geq r$ whenever $X_{\alpha} \neq X_{\alpha'}$. Then $\text{asdim } X \leq n$. 

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Union Theorem

The proof of the Lemma is based on the following:

**UNION THEOREM.** Let $X = \bigcup_{\alpha} X_{\alpha}$ be a metric space where the family $\{X_{\alpha}\}$ satisfies the inequality $\text{asdim } X_{\alpha} \leq n$ uniformly. Suppose further that for every $r$ there is a $Y_r \subset X$ with $\text{asdim } Y_r \leq n$ so that $d(X_{\alpha} \setminus Y_r, X_{\alpha'} \setminus Y_r) \geq r$ whenever $X_{\alpha} \neq X_{\alpha'}$. Then $\text{asdim } X \leq n$.

The family $\{X_{\alpha}\}$ of subsets of $X$ satisfies the inequality $\text{asdim } X_{\alpha} \leq n$ uniformly if for every $r < \infty$ one can find a constant $R$ so that for every $\alpha$ there exist $r$-disjoint families $\mathcal{U}_0^\alpha, \ldots, \mathcal{U}_n^\alpha$ of $R$-bounded subsets of $X_{\alpha}$ covering $X_{\alpha}$. 
Proof of Lemma

We prove that \( \text{asdim} \ AB \ldots A(B) \leq n \) by induction on the length of the product \( k \). The inequality is a true statement for \( k = 1 \). Assume that it holds for \( k \). Also assume that \( k \) is odd. Thus, \( \text{asdim} \ F_1 \ldots F_k \leq n \) where \( F_{2i-1} = A \) and \( F_{2i} = B \).

We show that \( \text{asdim} \ F_1 \ldots F_k B \leq n \). Consider the family \( \{wB \mid l(w) = k\} \). Since all sets \( wB \) are isometric to \( B \), \( \text{asdim} \ wB \leq k \) uniformly.

Given \( r \) we define \( Y_r = AB \ldots ACB_r \) where \( B_r \) is the \( r \)-ball in \( B \). One can show that \( d(wB \setminus Y_r, w'B \setminus Y_r) \geq r \) for \( w \neq w' \).

Then by the Union Theorem we obtain that \( \text{asdim}(F_1 \ldots F_k \cap L_k)B \leq n \) where \( L_k \) is the set of all elements \( w \in A*CB \) with \( l(w) = k \). The inequality \( \text{asdim}(F_1 \ldots F_m \cap L_{<m})B \leq n \) follows from induction assumption and the Finite Union Theorem. \( \square \)
The proof of the Action Theorem is based on the following:

**MAPPING THEOREM.** Let $\pi : Y \to X$ be Lipschitz and $\forall R > 0$, $\text{asdim} \pi^{-1}(B_R(x)) \leq k$ uniformly on $k$. Then $\text{asdim} Y \leq \text{asdim} X + k$. 

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Mapping Theorem

- The proof of the Action Theorem is based on the following:

  **MAPPING THEOREM.** Let $\pi : Y \rightarrow X$ be Lipschitz and $\forall R > 0$, $\operatorname{asdim} \pi^{-1}(B_R(x)) \leq k$ uniformly on $k$. Then $\operatorname{asdim} Y \leq \operatorname{asdim} X + k$.

- The Mapping Theorem does not always give a good estimate.

  **EXAMPLE:** Let $\pi : \mathbb{H}^2 \setminus B \rightarrow S$ be the geodesic projection of the complement to the horoball onto the horosphere. Then by the Mapping Theorem $\operatorname{asdim} \mathbb{H}^2 \setminus B \leq 1 + 2$. 

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We say that $(r, d) - \dim X \leq n$ if for every $r > 0$ there exists a $d$-bounded cover $\mathcal{U}$ of $X$ with $\text{ord} \mathcal{U} \leq n + 1$ and with the Lebesgue number $L(\mathcal{U}) > r$. We refer to such a cover as to $(r, d)$-cover of $X$. 
We say that $(r, d) - \dim X \leq n$ if for every $r > 0$ there exists a $d$-bounded cover $\mathcal{U}$ of $X$ with $\text{ord}\mathcal{U} \leq n + 1$ and with the Lebesgue number $L(\mathcal{U}) > r$. We refer to such a cover as to $(r, d)$-cover of $X$.

Clearly, $\text{asdim} X \leq n$ if for every $r > 0$ there is $d$ such that $X$ admits an $(r, d)$-cover.
Partition Theorem

PARTITION THEOREM. Let $X$ be a geodesic metric space. Suppose that for every $r > 0$ there is $d > 0$ and a partition $X = \bigcup_i^\infty W_i$ with $\operatorname{asdim} W_i \leq n$ uniformly such that $(r, d) - \dim \bigcup_i \partial W_i \leq n - 1$ for all $i$. Then $\operatorname{asdim} X \leq n$. 

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Partition Theorem

PARTITION THEOREM. Let $X$ be a geodesic metric space. Suppose that for every $r > 0$ there is $d > 0$ and a partition $X = \bigcup_{i}^{\infty} W_i$ with $\text{asdim } W_i \leq n$ uniformly such that $(r, d) - \dim \bigcup_{i} \partial W_i \leq n - 1$ for all $i$. Then $\text{asdim } X \leq n$.

In the above EXAMPLE the Partition Theorem gives right estimate $\text{asdim}(\mathbb{H}^2 \setminus B) \leq 2$ by taking a regular partition $\{F_i\}$ of $S = \mathbb{R}$ and taking the preimages $W_i = \pi^{-1}(F_i)$. 

Mapping Cylinder Theorem

For every $n \in \mathbb{N}$ there is a monotone tending to infinity function $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ with the following property: Given $\epsilon > 0$, let $W = N_\lambda(Y) \cup N_\lambda(Z)$ be the union of the $\lambda$-neighborhoods of $\lambda$-disjoint subsets $Y$ and $Z$ in a geodesic metric space $X$ with $\lambda \geq 8/\epsilon$. Then for every two covers $\mathcal{V}$ of $N_\lambda(Z)$ and $\mathcal{U}$ of $N_\lambda(Y)$ of the order $\leq n + 1$, and with $L(\mathcal{U}) > b(\mathcal{V}) > L(\mathcal{V}) \geq \mu(1/\epsilon)$, there is a $2b(\mathcal{U})$-cobounded $\epsilon$-Lipschitz map $f : W \to M_g$ to the mapping cylinder of a simplicial map $g : \text{Nerve}(\mathcal{V}) \to \text{Nerve}(\mathcal{U})$ between the nerves such that $f|_Z = p_\mathcal{V}|_Z$ where $p_\mathcal{V} : N_\lambda(Z) \to \text{Nerve}(\mathcal{V}) \subset M_g$ is the canonical projection.
PARTITION THEOREM. Let $X$ be a geodesic metric space. Suppose that for every $r > 0$ there is $d > 0$ and a partition $X = \bigcup_i^\infty W_i$ with $\text{asdim } W_i \leq n$ uniformly such that $(r, d) - \dim \bigcup_i \partial W_i \leq n - 1$ for all $i$. Then $\text{asdim } X \leq n$. 

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The idea of the proof. By means of the Mapping Cylinder Theorem an $\epsilon$-Lipschitz uniformly cobounded map can be constructed to the $n$-complex $K$ which is the union of the mapping cylinders of a simplicial maps induced by partial refinements $\mathcal{U} \prec \mathcal{W}_i$ of an $(r, d)$-cover of $\bigcup_i \partial W_i$ and $(2d, D)$-covers of $W_i$. 

On asymptotic dimension of Coxeter groups – p. 29/33
AMALGAMATION THEOREM.

\[ \text{asdim}(A *_C B) \leq \max\{\text{asdim } A, \text{asdim } B, \text{asdim } C + 1\} \]
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\[ \text{asdim}(A \ast_C B) \leq \max\{\text{asdim } A, \text{asdim } B, \text{asdim } C + 1\}. \]

Let \( n = \max\{\text{asdim } A, \text{asdim } B\} \). In view of the Weak Amalgamation Theorem we may assume that \( \text{asdim } C \leq n - 1 \). Given \( r > 0 \) we construct a partition of the Cayley graph of \( \Gamma \) into \( G_i \)s with \( (r, d) - \dim \bigcup_i \partial G_i \leq n - 1 \) and \( \text{asdim } G_i \leq n \). Then we apply the Partition Theorem.
**Amalgamation Theorem**

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We construct this partition using the action of \( \Gamma \) on the dual complex to the Bass-Serre tree.
The dual complex

The vertices of the dual complex $K$ are the left cosets $xC$. Two vertices $xC$ and $x'C$ are joined by an edge if and only if the edges in the Bass-Serre tree with these labels have a common vertex.
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$K$ is a tree-graded space in the sense of Drutu-Sapir with pieces $\Delta(A)$ and $\Delta(B)$, the 1-skeletons of the simplices spanned by $A/C$ or $B/C$. Thus, $K$ is partitioned into these pieces in a way that every two pieces have at most one common vertex and the nerve of the partition is a tree.
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The dual complex $K$ serves better because the action of $\Gamma$ on $K$ is transitive and the projection to the orbit $\pi : \Gamma \to K$, $\pi(g) = g(x_0)$, is 1-Lipschitz. Hence $\pi$ extends to a simplicial map of the Cayley graph. The corresponding projection for the Bass-Serre tree is 2-Lipschitz.
The Dual Complex
Amalgamation Theorem

We consider a partition of $K = \bigcup F_i$ into pieces taken from the proof of $(R, S) - \text{asdim } K \leq 1$ for sufficiently large $R$. Take the preimages $\pi^{-1}(F_i)$ and modify them into $G_i$, $i = 1, 2$. 
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THE END