Note Taker Checklist Form - MSRI

Name: Ioana Mihaila

E-mail Address/ Phone #: imihaia@csupomona.edu

Talk Title and Workshop assigned to:

Coxeter Groups & Artin Groups I/II

Intro to Geometric Group Theory

Lecturer (Full name): Ruth Charney

Date & Time of Event: 8/27/07 10:45 - 11:45 am
8/27/07 1 - 10 am

Check List:

( ) Introduce yourself to the lecturer prior to lecture. Tell them that you will be the note taker, and that you will need to make copies of their own notes, if any.

( ) Obtain all presentation materials from lecturer (i.e. Power Point files, etc). This can be done either before the lecture is to begin or after the lecture; please make arrangements with the lecturer as to when you can do this.

( ) Take down all notes from media provided (blackboard, overhead, etc.)

( ) Gather all other lecture materials (i.e. Handouts, etc.)

( ) Scan all materials on PDF scanner in 2nd floor lab (assistance can be provided by Computing Staff) – Scan this sheet first, then materials. In the subject heading, enter the name of the speaker and date of their talk.

Please do NOT use pencil or colored pens other than black when taking notes as the scanner has a difficult time scanning pencil and other colors.

Please fill in the following after the lecture is done:

1. List 6-12 lecture keywords: Coxeter Group, Artin Group, Deligne Complex, Salvetti Complex

2. Please summarize the lecture in 5 or less sentences:

Once the materials on check list above are gathered, please scan ALL materials and send to the Computing Department. Return this form to Larry Patague, Head of Computing (rm 214)

For Video Tapings-MSRI 9/2006
I Definitions & Examples

Coxeter graph

\[ \Gamma = \text{finite, simplicial, labelled graph} \]
vertices: \( S = \{ s_1, \ldots, s_n \} \)
edges: \[ \frac{m_{ij}}{s_i \rightarrow s_j} \quad m_{ij} \geq 2 \]
(convention: \( m_{ij} = \infty \) if no edge)

Coxeter group

\[ W = \langle S \mid s_i^2 = 1, (s_i s_j)^{m_{ij}} = 1 \rangle \]

= discrete linear gp generated by reflections

reflection (w.r.t. \( v \in \mathbb{R}^n \), \( \|v\| = 1 \)):
\[ r_v(w) = w - 2 \frac{v}{\langle v, w \rangle} v \quad \forall v \in \mathbb{R}^n \]

Example 1: \( n = 2 \):
\[ W = \langle s, t \mid s^2 = t^2 = 1, (st)^m = 1 \rangle = D_{2m} \]
if \( m = \infty \):
\[ W = \mathbb{Z}_2 \star \mathbb{Z}_2 = D_{\infty} \]

2) \( n = 3 \):
\[ W = \mathbb{T}(p_{1,3,3}) \]
reflect across walls of triangle

\[ T(2,3,3) \]

3) \( n \geq 5 \):
\[ W = \langle s_i, \ldots, s_n \mid s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \rangle \]
\( \text{"right-angled" Coxeter groups} \)

4) \( n > 2 \):
\[ W = \mathbb{Z} \cong \langle s_i, \ldots, s_n \rangle \quad \text{on} \ n \ \text{axes} \]
\[ H = \langle s_i \rangle \quad s_i = (\cdots) \]
Remark: can rewrite \( (s; s, i)^m = 1 \) as \( ss; . . . = s; s, i . . . \)

Artin group

\[ A_r = \langle S \mid \frac{ss; s, i . . . = s; s, i . . .}{s_i} \rangle \]

\[ = \pi_r (\text{hyperplane complement}) \text{ associated to } W_r \]

\( W \subset \mathbb{R}^n \) as reflection gp \( \rightarrow \) \( W \subset \mathbb{C}^n \)

\( Y = \text{non-singular pts} \) of this action

\[ = \mathbb{C}^n - \bigcup_{\text{reflection}} H_r \]

\( Y \rightarrow Y/W \) covering space

Thm: (Brieskorn '74) \( \pi_1 (Y/W) = A \)

Eg: \( W = \Sigma_n = \text{symm gp on } n \text{ letters} \)

\( \bigodot \mathbb{R}^n \) \( \rightarrow \) \( W \subset \mathbb{C}^n \) reflection hyperplanes: \( H_{ij} = \{ (z, . . ., z) \mid z_i = z_j \} \)

\( Y = \mathbb{C}^n - \bigcup H_{ij} \) config space of \( n \) distinct pts in \( \mathbb{C} \)

\( \pi_1 (Y/W) = \text{braid gp on } n \text{ strands} \)

\[ = \langle s_1, . . ., s_n \mid s_i s_j = s_j s_i, s = \rangle \]
Remark: All Artin gps are infinite, in fact they are conjectured to be torsion-free.

Example: \( T^2 \to A \xrightarrow{\gamma} A = \langle s, t \mid stst = tsts \rangle \Delta \) is big!

\( \Delta \) is central: \( \Delta (stst) = (tsts) t \), \( \text{Center}(A) = \langle \Delta \rangle \)

\( A/\langle \Delta \rangle = \langle x, y \mid x^2 = 1 \rangle = \mathbb{Z}_2 \ast \mathbb{Z} \)

Def: \( A_p \) is finite type (or spherical) if \( W_p \) is finite

Notation: \( T \leq S, W, A \)
II. Geometry: complexes associated to \( W_f, A_f \)

For \( T \subseteq S \)

\[
W_T = \text{subgrp \ of \ } \text{gen by} \ T = \text{Coxeter gp assoc to subgraph of} \ \Gamma \ \text{spanned by} \ T
\]

\[
\Gamma = \begin{array}{c}
\text{Diagram of \ } \Gamma \\
\end{array}
\]

\[
A_T = \text{subgrp of} \ A \ \text{gen by} \ T = \begin{array}{c}
\text{Diagram of} \ A_T
\end{array}
\]

Davis Complex for \( W \) (Tits, Davis)

\[
\mathcal{D}_W = \{ \{ wW_T \mid T \subseteq S, \ W_T \text{ finite} \} \}
\]

\[
\text{stab}(wW_T) = wW_T \text{ finite}
\]

\[
\mathcal{D}_W \text{ left mult, proper, cocompact}
\]

\[
K = \{ \{ W_T \mid T \subseteq S, \ W_T \text{ finite} \} \}
\]

\[
\mathcal{D}_W = W \times K / \sim
\]

\[
\text{Eq:} \ \Gamma = \begin{array}{c}
\text{Diagram of} \ \Gamma \\
\end{array}
\]

\[
W_T = \text{Euclidean triangle} \ \tilde{\delta}
\]

\[
\text{Remark: generators} \ s_i \ \text{act as "reflections" on} \ \mathcal{D}_W, \ \text{fixed set of reflection called "wall"}
\]

\[
K = \begin{array}{c}
\text{Diagram of} \ K \\
\end{array}
\]

\[
\mathcal{D}_W = \begin{array}{c}
\text{Diagram of} \ \mathcal{D}_W
\end{array}
\]
What's $\Delta_W$ good for?

1) Dictionary:

- **combinatorial prop of $\Delta_W$** \(\Rightarrow\) **geometric prop of $\Delta_W$**

- $w = s_i \cdots s_k$ word
- "gallery" in $\Delta_W$ from $k$ to $i$

- **minimum length word**
- gallery crosses each wall at most once

- **non-minimal**

- $w = s_i - s_j \cdots s_k$

- "Exchange Condition" \(\Rightarrow\) can shorten by reflection

2) nice geometry:

- **Davis**: $\Delta_W$ contractible

- **Moussong**: $\Delta_W$ CAT(0) \(\Rightarrow\) $W$ is CAT(0) \(\Leftrightarrow\) $W$ has no $\mathbb{Z}^2$ subgroup

- $\Delta_W$ hyperbolic \(\Leftrightarrow\) $W$ has a word hyperbolic

3) Interesting spaces in their own right.
Deligne complex for $A$ (Deligne, C-Dus, v.d.Dek)

$$\mathcal{D}_A = \left\{ a A_T \mid T \subseteq S, \ A_T \text{ finite type} \right\}$$

$$= A \times K/\sim$$

A and $\mathcal{D}_A$ stabilizers $a A_T a^{-1}$ are not finite

cocompact but not proper!

$\mathcal{D}_A$ not locally finite!

Salvetti complex for $A$ (Salvetti)

$$A \rightarrow W$$

has a set theoretic section:

we $w$, write a shortest word $w = s_{i_1} s_{i_2} \cdots s_{i_k}$

set $\sigma(w) = s_{i_1} s_{i_2} \cdots s_{i_k} \in A$

write $\hat{W} = \sigma(W)$, $\hat{W}_T = \sigma(W_T)$

$$\mathcal{S}_A = \left\{ a \hat{W}_T \mid T \subseteq S, \ W_T \text{ finite} \right\}$$

$\mathcal{S}_A$ cocompact, free!

$$\Sigma: \Gamma \xrightarrow{m} W = \operatorname{D}_{2m}$$

Exercise: $\mathcal{S}_A = \text{Cayley 2-complex for } A$

$\hat{W}_0 = \{ a_3 \}$

$\text{Ex} 3$
Topology of $X_A, s_A$

Fact: $\mathcal{X}_A \cong \tilde{Y} \cong s_A$. $\tilde{Y}$ = univ. cover of hyp. space.

Conj. $\tilde{Y}$ is contractible.

$\mathcal{X}_A \cong \tilde{Y} \cong s_A$ $\Rightarrow$ $Y/\pi \cong \tilde{Y} \cong Y/\pi$.

Conj $\Rightarrow Y/A$ is a $K(A, 1)$-space.

$\Rightarrow s_A/A$ is a (finite) $K(A, 1)$-space.

$\Rightarrow A$ is type $F$, torus-free.

* (Deligne) conj true for Amsler type $A$.
* (c. Davis) two metrics on $\mathcal{X}_A$
  * Mousterg metric: $\text{CAT}(0)$ if $s_A$ 2-dim'd (some)
  * Cubical metric: $\text{CAT}(0)$ $\iff$ $A$ is "FC-type"
* (Crisp): suff cond for $\text{CAT}(-1)$ metric $\Rightarrow A$ (weakly) rel hyp
Techniques & Open Questions

1. A finite type \((\Leftrightarrow W \text{ finite})\)

Powerful combinatorial techniques: Garside structure
(Garside, Dehornoy, Birman, Ko-Lee, Brady, Hess)

\(A^+ = \text{monoid of positive words}\)

A partial order on \(A^+\): \(a \preceq b\) if \(\exists c \in A^+, ac = b\).

Key facts:

- \(A^+\) is a lattice wrt \(\leq (\exists y \leq b\) and \(y \leq c\))
- \(\exists \Delta \in A^+, \forall A = A^+[\Delta]\)

Let \(M = \{a \in A^+ | a \preceq \Delta \} \supseteq S\).

Garside structure and canonical forms for \(A\) as words in \(M\).

\[ a \in A^+ : m_a = \gcd (a, \Delta) \Rightarrow a = m_a a_1 \]

\[ m_1 = \gcd (a, \Delta) \Rightarrow a = m_1 m_2 a_2 \]

\[ \Delta^m, m_1, m_2 \text{ canonical form} \]

\[ g \in A^+ \Rightarrow g = a \Delta^n, a \in A^+, n \text{ minimal} \]

\[ g = m_1 \cdots m_n \Delta^n \text{ canonical form} \]

Garside structure

- canonical forms for \(A\)
- bi-invariant structure on \(A\)
- \(G / A\) finite \(K(A)\)

\(G\) is \(A\) as \(H \rtimes \iota (\theta)\) group.
(2) **Affine type**

For \( \tilde{A}, \tilde{B}, \tilde{C} \), \( \text{conj. true} \).

\( A \leq \text{Mod(punctured sphere)} \)

Q: (McCullough) Is there an analogue of a Garside structure for affine type \( A \)?

(3) **Large type (\( \leq 2\text{-dim'} \))**

all \( m, j \geq 3 \) \( \Rightarrow \tilde{G}^A \) 2-dim'

**Combinatorial techniques ~ small techniques**

- Appel, Shap, Peifer, Johnson

**Geometric techniques**:

\( \tilde{G}^A \) CAT(0) \( \Rightarrow \) conj. true

Q: Does \( A \) act properly on a CAT(0) cubical complex?

(4) **FCC-type**

\( A \) finite type \( \Rightarrow \& \tilde{G}^A \) CAT(0) \( \Rightarrow \) \( m, j < \infty \) (no Euclidean or)

- (hyper or group)

**Combinatorial**

Normal forms with automatic structure

- Geometric: \( \tilde{G}^A \) CAT(0) \( \Rightarrow \) conj. true

Q: \( \tilde{G}^A \) is hyperbolic \( \Rightarrow \tilde{G}^A \) hyperbolic

(5) **RAAG's**

\( A \) is right-angled if \( m, j = 0 \) or big...