Note Taker Checklist Form - MSRI

Name:     jael Alegre-Kfir

E-mail Address/ Phone #: jael@math.utah.edu

Talk Title and Workshop assigned to: lattices acting on polyhedral complexes

Lecturer (Full name): Anne Thomas
Date & Time of Event: November 22, 2004 11:30 AM - 12:30 PM

Check List:

( ) Introduce yourself to the lecturer prior to lecture. Tell them that you will be the note taker, and that you will need to make copies of their own notes, if any.

( ) Obtain all presentation materials from lecturer (i.e. Power Point files, etc). This can be done either before the lecture is to begin or after the lecture; please make arrangements with the lecturer as to when you can do this.

( ) Take down all notes from media provided (blackboard, overhead, etc.)

( ) Gather all other lecture materials (i.e. Handouts, etc.)

( ) Scan all materials on PDF scanner in 2nd floor lab (assistance can be provided by Computing Staff) – Scan this sheet first, then materials. In the subject heading, enter the name of the speaker and date of their talk.

Please do NOT use pencil or colored pens other than black when taking notes as the scanner has a difficult time scanning pencil and other colors.

<table>
<thead>
<tr>
<th>Please fill in the following after the lecture is done:</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. List 6-12 lecture keywords:</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>2. Please summarize the lecture in 5 or less sentences.</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
</tbody>
</table>

Once the materials on check list above are gathered, please scan ALL materials and send to the Computing Department. Return this form to Larry Patague, Head of Computing (rm 214)

For Video Tapings - MSRI 9/2006
Lattices acting on polyhedral complexes

Angela Barnhill
Northwestern University

Anne Thomas
Cornell University
OUTLINE

1. Definitions and questions

2. Classical results

3. Lattices in automorphism groups of polyhedral complexes
   a) Davis-Moussong complexes
   b) right-angled buildings

4. Open questions
Basic definitions

- locally compact top. gp
- $\mu$ Haar measure

A subgroup $\Gamma \leq G$ is a lattice if

- $\Gamma$ is discrete
- $\mu(\Gamma \backslash G) < \infty$.

$\Gamma$ is uniform if $\Gamma \backslash G$ compact.
Questions

Given $G$, determine

1. *Existence*
   of unif./nonunif. lattices in $G$.
   How to construct?

2. *Covolumes*
   
   $\mathcal{V}(G) = \{ \mu(\Gamma \backslash G) \mid \Gamma \text{ lattice in } G \}$

   Is $\mathcal{V}(G)$ discrete?

3. *Commensurators* (with A. Bamhill)
   of unif. lattices in $G$.

   $\text{Comm}_G(\Gamma) = \{ g \in G \mid \Gamma \cap g\Gamma g^{-1} \text{ finite index in } \Gamma \text{ and } g\Gamma g^{-1} \}$

   Is $\text{Comm}_G(\Gamma)$ dense in $G$?
Classical results

Existence

A noncompact simple real Lie gp $\text{e.g. } \text{PSL}_n(\mathbb{R})$

Borel: $G$ admits unif. and nonunif. lattices

Margulis: in higher rank, all lattices arithmetic

Covolumes

$\text{e.g. } \text{PSL}_n(\mathbb{Q}_p)$

$+$ alg. gp over nonarch. local field

Borel: in higher rank, $\forall c > 0 \exists$ only finitely many lattices $\Gamma \leq H$ with $\mu(\Gamma \backslash H) < c$.

$\Rightarrow \forall (H) \text{ discrete.}$

Commensurators

Margulis: $\Gamma$ arithmetic $\iff \text{Comm}_c(\Gamma) \text{ dense in } G.$
Automorphism groups of polyhedral complexes

\[ X = \text{locally finite polyhedral complex} \]
\[ G = \text{Aut}(X) \quad \text{locally compact gp} \]

When is \( G \) nondiscrete?
If \( \exists g_n \in G, \ g_n \neq 1 \text{ fixing } \text{Ball}(n) \subseteq X \) then \( G \) is nondiscrete.

e.g. \( X = \text{locally finite tree} \)

Motivation:

Theorem (Tits) \( X = \text{building for } G \) higher rank alg. gp over nonarch.
local field e.g. \( G = \text{PSL}_3(\mathbb{Q}_p) \).
Then \( G \) finite index or cocompact in \( \text{Aut}(X) \).
Theorem (Serre)

Let \( G = \text{Aut}(X) \), \( X \) locally finite poly. ex.

If \( G \backslash X \) compact then Haar measure \( \mu \) on \( G \) may be normalised s.t. for all discrete \( \Gamma \leq G \)

\[
\mu(\Gamma \backslash G) = \sum_{x \in V(\Gamma \backslash X)} \frac{1}{|\Gamma_x|}
\]

hence:

\( \Gamma \) a lattice \( \iff \) series converges.

Moreover:

\( \Gamma \) a unif. lattice \( \iff \Gamma \backslash X \) compact.

go to board
Existence and covolumes of lattices for Davis-Moussong complexes

**Theorem (T)**

Let $X = X_W$ be the Davis-Moussong complex for a Coxeter group $W$.

Let $G = \text{Aut}(X)$.

Suppose

- $G$ is nondiscrete (Haglund-Paulin)
- 2 more technical conditions hold (If $\dim(X) = 2$, enough that all $m_{ij} = m$ even, and that $\text{Aut}(L)$ transitive on vertices of $L$)

Then $G$ admits

1. a nonuniform lattice $\Gamma$
2. an infinite family of uniform lattices $(\Gamma_n)$

s.t. $\mu(\Gamma_n \backslash G) \rightarrow \mu(\Gamma \backslash G)$.

Hence $\mu(G)$ is nondiscrete.
Density of commensurators for right-angled buildings

**Theorem (Bamhili-T, Haglund)**
Let $X$ be a right-angled building s.t. $G = \text{Aut}(X)$ nondiscrete.
Let $\Gamma_0 \leq G$ be the "standard unit lattice".
Then $\text{Comm}_G(\Gamma_0)$ is dense in $G$.

**Corollary**
If $\text{dim}(X) = 2$, $\text{Comm}_G(\Gamma)$ dense in $G$ for all unit lattices $\Gamma \leq G$.

**Proof** $\Gamma$ commens. to $\Gamma_0$ by thm of Haglund.
Coxeter groups + Davis-Moussong cx s

\[ W = \langle s_i, i \in I \mid (s_i s_j)^{m_{ij}} = 1 \rangle \]

\[ m_{ii} = 2, \ m_{ij} \geq 2 \] for \( i \neq j \)

Let \( L \) be the finite nerve of \( W \), i.e., simplicial cx with

* vertices \( s_i \)

* edges \( s_i \leadsto s_j \) if \( m_{ij} < \infty \)

\[ \iff W_{[i,j]} = \langle s_i, s_j \rangle \text{ is finite} \]

* 2-simplices

\[ \iff W_J = \langle s_i, s_j, s_k \rangle \text{ finite} \]

for \( J = \{ i, j, k \} \subseteq I \)

* etc

Spherical subgroups of \( W \) are

\[ W_J = \langle s_j, j \in J \subseteq I \rangle \] which are finite.
Davis-Moussong complexes

N Coxeter gp, \( L \) = finite nerve of \( W \)

Let \( C(L) \) be the cubical cone on \( L \)

(Davis-Januszkiewicz-Scott)

e.g. \( L = s_3 \triangleleft s_2 \triangleleft s_1 \)

\[ \begin{align*}
C(L) = \\
\end{align*} \]

\( n \)-simplex in \( L \) \( \leftrightarrow \) \((n+1)\)-cube in \( C(L) \)
Davis-Moussong complex

\[ X = C(L) \times \mathbb{Z}^{s} \]

Glue together \( n \)-many copies of \( C(L) \) along faces.

Links of vertices in \( X \) are \( L \).

\( V \) acts on \( X \) properly discontinuously and cocompactly.

\[ \Rightarrow W \text{ uniform lattice in Aut}(X). \]
Vondiscreteness of $\text{Aut}(X)$

$X$ Davis-Moussong complex for $W$
$L =$ finite nerve of $W =$ links of vertices in $X$.

**Theorem (Haglund-Paulin)**

$\text{Aut}(X)$ nondiscrete

$\Rightarrow \exists \ g$ in group of label-preserving automorphisms of $L$ s.t. $g$ fixes star of some $v \in \text{Vert}(L)$ and $g \neq 1$.

E.g.

If $m_{24} = m_{34}$ and $m_{25} = m_{35}$ then $\text{Aut}(X)$ is nondiscrete.
Construction of sequence of uniform lattices in Aut(X)

e.g. \( L = \)

Think of \( W \) as fundamental group of complex of groups over cubical cone \( C(L) \):

- \( W_{s, s_i}, s_{i+1} \cong D_{2m} \)
- \( W_{s, s_i} = \langle s_i \rangle \cong \mathbb{Z}/2\mathbb{Z} \)
- \( W_{s, s_i} \cong \mathbb{Z}/2\mathbb{Z} \)
- \( W_{s, s_i} \cong \mathbb{Z}/2\mathbb{Z} \)
- \( W_{s, s_i} \cong D_{2m} \)

Face groups
Edge groups
Vertex groups
The lattice $L$

$\text{Aut}(X)$ nondiscrete

$\Rightarrow \exists g \in \text{Aut}(L)$ (label-preserving)

s.t. $g$ fixes star of say $s_i \in \text{Vert}(L)$

Wlog $\langle g \rangle | = q$ prime, so for all $s_i$,

$\text{Stab}_{\langle g \rangle} (s_i) = \{1\} \text{ or } \langle g \rangle$.

Action of $g$ extends to $C(L)$ and to complex of groups for $W$.

Get $\Gamma'_1 = \text{fund. gp of resulting complex of groups}$.
The lattice $\Gamma_2$

$g$ acts on purple

using technical conditions

$g$ acts on red

fund. gp. $W$

covolume $\frac{1}{4} + \frac{1}{4}$
The lattices \((\Gamma_n)\)

- \(\Gamma_n\) represents the lattices with a "rooted tree" depth \(n\).
- The fundamental group is denoted as \(\Gamma_n\).
- The volume is given by \(\frac{1}{q} + \frac{1}{q^2} + \cdots + \frac{1}{q^n}\).

Coverings \(\Rightarrow\) \(\Gamma_n \leq \text{Aut}(X)\)

Uniform lattice

Remark: all \(\Gamma_n\) commensurate to \(W\); for some 2-dim \(X\), all uniform lattices are commensurate to \(W\) (Haglund).

Moussong's conditions for \(X\) CAT(-1).

Get new hyperbolic groups, including dim \(\geq 2\).
Uniform lattice $\Gamma$ implies infinite \textit{one-ended} tree. Find $\gamma_1$, infinite-sheeted cover $\widetilde{M} = \mathbb{R}^2$.

\[ \text{fund. gr. } \frac{1}{8} \leq \frac{1}{q} \leq \frac{1}{2} \]

Coverings $\Gamma \leq \text{Aut}(\mathbb{X})$ nonuniform lattice.

Remark: expect that variations in construction give many noncommens.

Nonuniform lattices, using invariants of Conell-Hruska.
light-angled buildings

Coxeter group $W$ is right-angled if all $m_{ij} = 2$ or $\infty$.

\[ W = \langle s_1, \ldots, s_p \mid s_i^2 = (s_i s_{i+1})^3 = 1 \rangle \]

A right-angled building has apts. Davis-Moussong complexes for $W$.

\[ \text{e.g. Bourdon's building } I_{p,q} \]

Apts modelled on $W$ above $q = \text{degree of branching at codim 1 faces}$

$\text{Aut}(X)$ nondiscrete $\iff X$ is "thick" building

\[ \text{e.g. } q \geq 3 \]
I_{6,2}
light-angled buildings

Some "higher-rank" behaviour:

Theorem (Bourdon–Pajot)
$I_{p,q}$ is quasi-isometrically rigid.

BUT "tree-like" results too:

Haglund

$T$

Barnhill–T
Density of commensurators for right-angled buildings

**Theorem (Haglund, Barnhill-T)**

Let $X$ be a right-angled building.
Let $\Gamma_0 \leq \text{Aut}(X)$ be the "standard unif. lattice".
Then $\text{Comm}_c(\Gamma_0)$ is dense in $G = \text{Aut}(X)$.

**Theorem (Bass-Kulkami, Liu)**

Let $T$ be a locally finite tree.
Let $\Gamma_0 \leq \text{Aut}(T)$ be "standard unif. lattice".
Then $\text{Comm}_c(\Gamma_0)$ is dense in $G = \text{Aut}(T)$.

(we give new simpler proof for $T$ regular or biregular tree.)
The standard unit lattice $\Gamma_0$

Fund gp of complex of finite cyclicgps and direct products of these.

e.g. $\mathbb{Z}/q, \mathbb{Z}/q^2, \mathbb{Z}/q^3, \mathbb{Z}/q^4, \mathbb{Z}/q^5$

Key property: $G_0 = \text{Aut}_o(X)$ type-preserving automs.

$$\Gamma_0 \backslash X = G_0 \backslash X.$$ 

$$\Rightarrow \forall \ x \in X$$

$$G_0 = \text{Comm}_{G_0}(\Gamma_0) \cdot \text{Stab}_{G_0}(x)$$

$\Rightarrow$ enough to show

$$\text{Comm}_{G_0}(\Gamma_0)$$ dense in $\text{Stab}_{G_0}(x)$. 
Want to show:
\[ \forall g \in \text{Stab}_{G_0}(x) \]
\[ \forall k \geq 1 \]
there is a \( \gamma \in \text{Comm}_{G_0}(\Gamma_0) \) s.t.
\[ g \mid B_x(k) = \gamma \mid B_x(k) . \]

Steps
- \( g \) fixes some ball \( B_x(r) \) for \( 1 \leq r \leq k \)
- Analyse action of \( g \) on \( B_x(r+1) \):
  - show can write \( g \) as product of transpositions (simpler automorphisms)
  - approximate transpositions by elements of lattices commens. to \( \Gamma_0 \)
  - (construct covering of complexes of groups)
Transpositions

$X = \text{tree}$

- Transposition
  - fixes $B_x(r)$
  - switches 2 edges in $B_x(r+1)$

$X = \text{right-angled building}$

- Switching 2 faces in $B_x(r+1)$
  entails switching adjacent faces

- Use product structure of $\text{Aut}(\text{links})$. 