Normal automorphisms of relatively hyperbolic groups

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Let G be a group. Consider the following subgroups of the automorphism group of G:

- The subgroup of normal automorphisms

$$\mathrm{Aut}_n(G) = \{\alpha \in \mathrm{Aut}(G) | \alpha(N) = N \text{ for all normal subgroups N in } G\}$$

- The subgroup of conjugating automorphisms (a.k.a. pointwise inner automorphisms):

$$\mathrm{Aut}_c(G) = \{\alpha \in \mathrm{Aut}(G) | \forall g \ \exists t = t(g) \in G \text{ such that } \alpha(g) = t^{-1}gt\}$$

  Remark. *If $t(g)$ doesn't depend on g then α is conjugation by t but in general t may vary.*

We have:

$$\mathrm{Inn}(G) \lhd \mathrm{Aut}_c(G) \lhd \mathrm{Aut}_n(G) \lhd \mathrm{Aut}(G)$$

But how "bad" can the quotients be?

- *Minasyan*: For any countable group Q there exists a group G with two conjugacy classes such that $\mathrm{Out}(G) \cong Q$.
  Since there are only two conjugacy classes (the identity and the rest) all automorphisms are piecewise inner therefore $\mathrm{Aut}_c(G)/\mathrm{Inn}(G) = \mathrm{Out}(G) \cong Q$.

- *Endimiouni*: G is free nilpotent of class c then:

  $c = 2$ $\mathrm{Aut}_n(G) = \mathrm{Inn}(G)$

  $c \geq 3$ $\mathrm{Aut}_c(G) = \mathrm{Inn}(G)$ and $|\mathrm{Aut}_n(G)/\mathrm{Aut}_c(G)| = \infty$

- *Ramankov*: If G is free solvable (i.e. $G \cong F_m/F_m^{(c)}$ where $F_m^{(c)}$ is the cth term of the derived series) of class $c \geq 2$ then $\mathrm{Aut}(G)_n = \mathrm{Inn}(G)$

Question: When does $\mathrm{Aut}_n(G) = \mathrm{Inn}(G)$ or $\mathrm{Aut}_c(G) = \mathrm{Inn}(G)$?
These questions are motivated by applications to $\mathrm{Out}(G)$

Theorem 1 (Baumslag). *If G is finitely generated and risidually finite then $Aut(G)$ is residually finite.*

This theorem does not have an analogue for $\mathrm{Out}(G)$. Indeed:

Theorem 2 (Wise). *For any finitely presented group Q there exists a risidually finite G such that $Q \hookrightarrow Out(G)$*

Thus if Q is an infinite simple group $\mathrm{Out}(G)$ would contain an infinite simple group and could not be risidually finite.

Definition 3. *G is conjugacy separable if for all non conjugate g, h there exists a homomorphism $\varepsilon : G \to K$ where K is finite such that $\varepsilon(g), \varepsilon(h)$ are not conjugate in K.*

Theorem 4 (Grossman). *If G is finitely generated and conjugacy separable then $Aut(G)/Aut_c(G)$ is risidually finite.*

Corollary 5. *If G is finitely generated and conjugacy separable and $Aut_c(G) = Inn(G)$ then $Out(G)$ is risidually finite.*

1. *Grossman*: If $G = F_m$ or $G = \pi_1(S)$ where S is closed and oriented then $\mathrm{Aut}_c(G) = \mathrm{Inn}(G)$, hence $\mathrm{Out}(F_m)$ and $\mathrm{Mod}(S_g)$ are residually finite.

2. *Allenby, Kim, Tang*: $G = H *_C K$ where H, K are free and C is maximal cyclic in both H and K then $\mathrm{Aut}_c(G) = \mathrm{Inn}(G)$ and by Dyer G is conjugacy separable.

   Example. *$G = < x_1, \dots, x_k | x_1^2 \cdots x_k^2 = 1 >$ for $k \geq 4$ then $Out(G)$ is risidually finite.*

3. *Metaftsis and Sikiotis*: G is a non-elementary relatively hyperbolic (hyperbolic relative to a proper non-cyclic subgroup) group then $|\mathrm{Aut}_c(G)/\mathrm{Inn}(G)| < \infty$ (this is not enough to conclude $\mathrm{Out}(G)$ is risidually finite).
   If G is conjugacy separable $\Longrightarrow \mathrm{Out}(G)$ is finite-by-r.f. (there are many examples of such groups which are not risidually finite).

4. *Liubotzky*: $\forall m \geq 2$ $\mathrm{Aut}_n(F_m) = \mathrm{Inn}(F_m)$.

5. *Neshadim*: $G = A * B$ then $\mathrm{Aut}_n(G) = \mathrm{Inn}(G)$

6. *Bogopolskii, Kudryavtseva, Zeischang*: If $g \geq 2$ then $\mathrm{Aut}_n(\pi_1(S_g)) = \mathrm{Inn}(\pi_1(S_g))$

The following example shows that result 6 doesn't carry through to realtively hyperbolic groups.

Example (Robinson). *For every finite group K there exists a finite group L such that $K \hookrightarrow Aut_n(L)/Inn(L)$. Therefore, if $G = F_2 \times L$ then $K \hookrightarrow Aut_n(G)/Inn(G)$.*

Definition 6. *G is non-elementary relatively hyperbolic. The finite core of G, $K(G)$ is the (unique) maximal finite normal subgroup of G.*

Theorem 7 (Osin). *For any non-elementary relatively hyperbolic group G, $Inn(G)$ has finite index in $Aut_n(G)$. If in addition, $K(G) = \{1\}$ (for example if G is torsion free) then $Aut_n(G) = Inn(G)$. Moreover, in the latter case $\exists R \lhd G$ such that $\forall \alpha \in Aut(G)$ $\alpha(R) = R \Longrightarrow \alpha \in Inn(G)$*

Corollary 8. *If G is relatively hyperbolic and conjugacy seperable then $Out(G)$ is risidually finite.*

Proof of the theorem : G is hyperbolic relative to $\{H_\lambda\}_{\lambda\in\Lambda}$ where H_λ is proper and not virtually cyclic. First assume $K(G) = \{1\}$. g is called hyperbolic if it's of infinite order and cannot be conjugated into any H_λ (otherewise: finite order elements are called elliptic, elements that can be conjugated into an H_λ are parabolic).

Lemma 9 (D.O.). *Let $g \in G$ be hyperbolic then*

1. *$\exists!$ maximal elementary virtually cyclic subgroup $E(g) \leq G$ that contains g.*

2. *G is hyperbolic relative to $\{H_\lambda\}_{\lambda\in\Lambda} \cup \{E(g)\}$.*

Lemma 10 (D.O.). *If $\alpha \in Aut(G) \setminus Inn(G)$ then there exists a hyperbolic element $g \in G$ such that $\alpha(g)$ is hyperbolic and not conjugate to an element of $E(g)$.*

Definition 11. *Let G be generated by the finite set X, and $H \leq G$ then $l(H) = \min\limits_{h\in H\setminus\{1\}} |h|_X$*

Theorem 12 (Groves - Manning - O.). *(Hyperbolic Dehn Surgery) If G is hyperbolic rel. $\{H_\lambda\}_{\lambda\in\Lambda}$ then there exists a $d > 0$ such that $\forall N_\lambda \lhd H_\lambda$ with $l(N_\lambda) > d$ we have:*

- *$N \cap H_\lambda = N_\lambda$ where $N = <\cup N_\lambda>^G$ ($H_\lambda/N_\lambda \hookrightarrow G/N$).*
- *G/N is hyperbolic rel to $\{H_\lambda/N_\lambda\}_{\lambda\in\Lambda}$*

We wish to show that $\mathrm{Aut}_n(G) = \mathrm{Inn}(G)$. Take $\alpha \in \mathrm{Aut}(G)/\mathrm{Inn}(G)$ and $g \in G$ given by lemma 10. Without loss of generality, $< g > \lhd E(g) \implies < g^n > \lhd E(g)$ for all n. Let $h = \alpha(g)$ and observe that G is hyperbolic rel. $\{H_\lambda\} \cup \{E(g), E(h)\}$. Consider normal subgroups $\{1\} \lhd H_\lambda$, $< g^n > \lhd E(g)$, $\{1\} \lhd E(h)$ and notice that $l(< g^n >) > d$ when $n >> 1$. Applyind the Dehn surjury theorem to this collection, we get $N =< g^n >^G$ such that

$$N \cap E(h) = \{1\} \cap E(h) = \{1\}$$
$$h^n = \alpha(g^n) \in \alpha(N) \cap E(h)$$

Get $\alpha(N) \neq N$ which is a contradiction.

If $K(G) \neq \{1\}$ then $K(G/K(G)) = \{1\}$ and one can apply the theorem to the quotient and transmit the result to G up to finite index. □

Let G be a group, $\mathcal{H}$ the collection of all normal subgroups N of G such that G/N is torsion free hyperbolic. Set

$$Aut_n(G, \mathcal{H}) = \{\alpha \in Aut(G) \mid \alpha(N) = N \ \forall\, N \in \mathcal{H}\}.$$

From the proof of the main theorem we also obtain

Corollary 13. *If G is non-elementary torsion free hyperbolic, then $Aut_n(G, \mathcal{H}) = Inn(G)$.*

Question. *Suppose G is hyperbolic and torsion free and R is a finitely generated non-elementary subgroup of G. Does it follow that $Aut_n(R, \mathcal{H}) = Inn(R)$?*

Applying small cancellation theory it is easy to show that such an R is residually torsion free hyperbolic. If the answer is yes then there exists a non-residually finite hyperbolic group.