1. (a) What is the norm of the partition $P = \{0, \pi/8, \pi/4, 3\pi/8, \pi/2\}$?

The norm is the width of the largest subinterval, which is $\pi/8$.

(b) Use the partition $P$ and some of the values from the following table to write down a Riemann sum approximation for $\int_0^{\pi/2} \sin x \, dx$ (you don’t need to simplify it – you can leave it as a sum of fractions).

<table>
<thead>
<tr>
<th>$x$</th>
<th>$0$</th>
<th>$\pi/6$</th>
<th>$\pi/4$</th>
<th>$\pi/3$</th>
<th>$\pi/2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sin x$</td>
<td>$0$</td>
<td>$1/2$</td>
<td>$1/\sqrt{2}$</td>
<td>$\sqrt{3}/2$</td>
<td>$1$</td>
</tr>
</tbody>
</table>

The points in the partition $P$ give the endpoints of the bases of 4 rectangles whose area we need to sum. We need to choose a value $c_k$ in each subinterval to determine the height of each rectangle. For example let $c_1 = 0$, $c_2 = \pi/6$, $c_3 = \pi/4$ and $c_4 = \pi/2$. This gives the Riemann sum

$$
\int_0^{\pi/2} \sin x \, dx \approx S = \sum_{k=1}^4 \sin(c_k) \Delta x_k = (0 + 1/2 + 1/\sqrt{2} + 1)\pi/8.
$$

2. Evaluate the following integrals.

(a) $\int_{-2}^{2} (x^3 - 2x + 3) \, dx = \left[ \frac{x^4}{4} - x^2 + 3x \right]_{-2}^{2}
= \left( \frac{2^4}{4} - 2^2 + 3(2) \right) - \left( \frac{(-2)^4}{4} - (-2)^2 + 3(-2) \right)
= 12.$

(b) $\int_{1}^{2} \sqrt{3x+1} \, dx$

Substitute $u = 3x + 1$, $du = 3dx$, and change the limits: $x |_{1}^{2} \rightarrow u |_{4}^{7}$.

The integral becomes

$$
\frac{1}{3} \int_{4}^{7} u^{\frac{1}{2}} \, du = \left[ \frac{2}{9} u^{\frac{3}{2}} \right]_{4}^{7} = \frac{2}{9} \left( 7^{\frac{3}{2}} - 8 \right).
$$

(c) $\int x^3 \cos(x^4 + 2) \, dx$

Substitute $u = x^4 + 2$, $du = 4x^3 \, dx$. The integral becomes

$$
\frac{1}{4} \int \cos u \, du = \sin u + C = \frac{\sin(x^4 + 2)}{4} + C.
$$
3. How many subintervals are required to estimate \( \int_0^3 \sin x \, dx \) to within an absolute error of 0.02 using

(a) the trapezoid rule

Use the trapezoid rule error estimate:

\[
|E_T| \leq \frac{b - a}{12} h^2 M,
\]

where \( a = 0, b = 3, h = \frac{b-a}{n} = \frac{3}{n}, \) and \( M = 1 \) is an upper bound for the absolute value of the second derivative of \( \sin x \). This means we need to find \( n \) with

\[
\frac{3}{12} \left( \frac{3}{n} \right)^2 \leq 0.02 = \frac{1}{50} \iff n^2 \geq \frac{3^3 \cdot 50}{12} = \frac{3^2 \cdot 25}{2} = 107.5.
\]

Since \( 10^2 = 100 \) and \( 11^2 = 121 \) we see that we need to 11 subintervals will suffice.

(b) Simpson’s rule.

Using the Simpson’s rule error estimate formula:

\[
|E_S| \leq \frac{b - a}{180} h^4 M,
\]

where \( a = 0, b = 3, h = \frac{b-a}{n} = \frac{3}{n}, \) and \( M = 1 \) is an upper bound for the absolute value of the fourth derivative of \( \sin x \). This means we need to find \( n \) with

\[
\frac{3}{180} \left( \frac{3}{n} \right)^4 \leq 0.02 = \frac{1}{50} \iff n^4 \geq \frac{3^5 \cdot 50}{180} = \frac{3^3 \cdot 5}{2} = 67.5.
\]

Since \( 2^4 = 16 \) and \( 3^4 = 81 \) and the number of subintervals has to be even for Simpson’s rule, we see that 4 subintervals are required.

4. Find the volume of the solid generated by rotating the region bounded by the curves \( y = \sqrt{x}, y = x \)

(a) about the \( x \)-axis

(b) about the line \( x = -1 \).

For both parts of this question you have a choice between the washer and shell methods.
(a) Using the washer method, the volume is
\[
\pi \int_0^1 (\sqrt{x})^2 - x^2 \, dx = \pi \int_0^1 x - x^2 \, dx
\]
\[
= \pi \left[ \frac{x^2}{2} - \frac{x^3}{3} \right]_0^1
\]
\[
= \pi \left( \frac{1}{2} - \frac{1}{3} \right) = \frac{\pi}{6}.
\]

(b) Rewriting the curves as \( x = y^2 \) and \( x = y \) and using the washer method, the volume is
\[
\pi \int_0^1 (y + 1)^2 - (y^2 + 1)^2 \, dy = \pi \int_0^1 -y^4 - y^2 + 2y \, dy
\]
\[
= \pi \left[ -\frac{y^5}{5} - \frac{y^3}{3} + y^2 \right]_0^1
\]
\[
= \pi \left( -\frac{1}{5} - \frac{1}{3} + 1 \right) = \frac{7\pi}{15}.
\]

5. Find the length of the curve \( y = \int_0^x \sqrt{\cos 2t} \, dt \) from \( x = 0 \) to \( x = \pi/4 \).

Use the arclength formula
\[
L = \int_a^b \sqrt{1 + [f'(x)]^2} \, dx.
\]
We have \( f(x) = \int_0^x \sqrt{\cos 2t} \, dt \), so by the fundamental theorem of calculus
\[
f'(x) = \sqrt{\cos 2x}.
\]
Using the trig identities \( \cos 2x = \cos^2 x - \sin^2 x \) and \( 1 = \cos^2 x + \sin^2 x \) we find that
\[
\sqrt{1 + [f'(x)]^2} = \sqrt{1 + \cos 2x} = \sqrt{2} \cos^2 x = \sqrt{2} |\cos x|.
\]
Then since \( \cos x \) is positive on \([0, \pi/4]\),
\[
L = \sqrt{2} \int_0^{\pi/4} \cos x \, dx = \sqrt{2} (\sin(\pi/4) - \sin 0) = 1.
\]

6. Use the method of slicing to find the volume of the region in the first octant (i.e. \( x \geq 0, y \geq 0, z \geq 0 \)) which is bounded by the coordinate planes \( x = 0, y = 0, z = 0 \) and by the plane \( x + y + z = 1 \). (HINT: the region has 4 triangular faces and vertices at the points \((0,0,0), (1,0,0), (0,1,0), (0,0,1)\).)

Picture the region as in the hint. The cross-sections perpendicular to the \( z \)-axis are right isosceles triangles with side \( z - 1 \) and area
\[
A(z) = \frac{1}{2} (z - 1)^2.
\]
The volume is then given by
\[
\int_0^1 A(z) \, dz = \frac{1}{2} \int_0^1 (z - 1)^2 \, dz
\]
\[
= \frac{1}{2} \int_{-1}^0 u^2 \, du
\]
\[
= \frac{1}{2} \left[ \frac{u^3}{3} \right]_{-1}^0
\]
\[
= \frac{1}{6},
\]
where we use the substitution \( u = z - 1 \) in the second line.