1. We use the Simpson’s rule error estimate formula:

\[ |E_s| \leq \frac{b-a}{180} h^4 M, \]

where \( a = 0, \ b = 3, \ h = \frac{b-a}{n} = \frac{3}{n}, \) and \( M = 1 \) is an upper bound for the absolute value of the fourth derivative of \( \sin x \). This means we need to find \( n \) with

\[
\frac{3}{180} \left( \frac{3}{n} \right)^4 \leq 0.02 = \frac{1}{50},
\]

\[ \iff \quad n^4 \geq \frac{3^5 \cdot 50}{180} = \frac{3^3 \cdot 5}{2} = 67.5. \]

Since \( 2^4 = 16 \) and \( 3^4 = 81 \) and the number of subintervals has to be even for Simpson’s rule, we see that 4 subintervals are required.

2. For both parts of this question you have a choice between the washer and shell methods.

(a) Using the washer method, the volume is

\[
\pi \int_0^1 (\sqrt{x})^2 - x^2 \, dx = \pi \int_0^1 x - x^2 \, dx = \pi \left[ \frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \pi \left( \frac{1}{2} - \frac{1}{3} \right) = \frac{\pi}{6}.
\]

(b) Rewriting the curves as \( x = y^2 \) and \( x = y \) and using the washer method, the volume is

\[
\pi \int_0^1 (y + 1)^2 - (y^2 + 1)^2 \, dy = \pi \int_0^1 -y^4 - y^2 + 2y \, dy = \pi \left[ -\frac{y^5}{5} - \frac{y^3}{3} + y^2 \right]_0^1 = \pi \left( -\frac{1}{5} - \frac{1}{3} + 1 \right) = \frac{7\pi}{15}.
\]

3. (a) Use the arclength formula:

\[
F(x) = \int_0^x \sqrt{1 + f'(t)^2} \, dt = \int_0^x \sqrt{1 + 4t^2 \sinh^2(t^2)} \, dt.
\]

(b) By the Fundamental Theorem of Calculus,

\[
F'(x) = \sqrt{1 + 4x^2 \sinh^2(x^2)}
\]

\[ \iff F'(1) = \sqrt{1 + 4 \sinh^2 1}. \]

4. Since the population is growing exponentially, it is given by a function \( P(t) = P_0 e^{rt} \), where \( r \) is a positive constant and \( P_0 \) is the initial population. The given data tell
us that
\[ 300 = P_0 e^r \]
\[ 900 = P_0 e^{2r}. \]
Dividing (2) by (1) yields \( e^r = 3 \), and substituting this in either equation gives \( P_0 = 100 \).

5. (a) Rewrite the integral as \( \int e^{2x+1} \cos(e^{2x}) \, dx = e \int e^{2x} \cos(e^{2x}) \, dx \). Let \( u = e^{2x} \) so that \( du = 2e^{2x} \, dx \). Then we have
\[
e \int e^{2x} \cos(e^{2x}) \, dx = \frac{e}{2} \int \cos u \, du = \frac{e}{2} \sin u + C = \frac{e}{2} \sin(e^{2x}) + C.
\]
(b) Solve using integration by parts. Let \( u = e^x \) and \( dv = \cos x \, dx \) so that \( du = e^x \, dx \) and \( v = \sin x \). Therefore,
\[
\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx.
\]
Now use integration by parts to evaluate \( \int e^x \sin x \, dx \). Let \( u = e^x \) and \( dv = \cos x \, dx \) so that \( du = e^x \, dx \) and \( v = -\cos x \). Therefore,
\[
\int e^x \cos x \, dx = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx + C',
\]
\[
2 \int e^x \cos x \, dx = e^x \sin x + e^x \cos x + C',
\]
\[
\int e^x \cos x \, dx = \frac{1}{2} e^x \sin x + \frac{1}{2} e^x \cos x + C.
\]
(c) Use the trigonometric substitution \( x = 2 \sin \theta \). Therefore, \( dx = 2 \cos \theta \, d\theta \) and we have
\[
\int \sqrt{4 - x^2} \, dx = \int \left( \sqrt{4 - 4 \sin^2 \theta} \right) (2 \cos \theta) \, d\theta,
\]
\[
= \int (2 \cos \theta)(2 \cos \theta) \, d\theta = 4 \int \cos^2 \theta \, d\theta,
\]
\[
= 2 \int (1 + \cos 2\theta) \, d\theta = 2\theta + \sin 2\theta + C.
\]
Using the fact that \( \theta = \sin^{-1} \frac{x}{2} \) and \( \sin 2\theta = 2 \sin \theta \cos \theta = \frac{1}{2} x \sqrt{4 - x^2} \) (using an appropriate right triangle), the integral in terms of \( x \) is
\[
\int \sqrt{4 - x^2} \, dx = 2 \sin^{-1} \frac{x}{2} + \frac{1}{2} x \sqrt{4 - x^2} + C.
\]
6. (a) Use the substitution \( u = 1 + \ln x \). Then \( du = \frac{1}{x} \, dx \) and we have
\[
\int_1^{e^2} \frac{1}{x(1 + \ln x)^2} \, dx = \int_1^3 \frac{1}{u^2} \, du = \left[ -\frac{1}{u} \right]_1^3 = -\frac{1}{3} + 1 = \frac{2}{3}.
\]
(b) The integral is improper because the integrand is undefined at the upper limit. Therefore, we evaluate
\[
\int_0^3 \frac{1}{\sqrt{3-x}} \, dx = \lim_{b \to 3^-} \int_0^b \frac{1}{\sqrt{3-x}} \, dx.
\]
Let \( u = 3 - x \) so that \( du = -dx \) and the integral becomes
\[
\int_0^b \frac{1}{\sqrt{3-x}} \, dx = \int_3^{3-b} \frac{-1}{\sqrt{u}} \, du = [-2\sqrt{u}]_3^{3-b} = -2\sqrt{3-b} + 2\sqrt{3}.
\]
Then,
\[
\lim_{b \to 3^-} \int_0^b \frac{1}{\sqrt{3-x}} \, dx = \lim_{b \to 3^-} -2\sqrt{3-b} + 2\sqrt{3} = 2\sqrt{3}.
\]
(c) Here, the integrand has an asymptote at \( x = 1 \). Therefore, we split the integral in two:
\[
\int_0^2 \frac{1}{(1-x)^2} \, dx = \int_0^1 \frac{1}{(1-x)^2} \, dx + \int_1^2 \frac{1}{(1-x)^2} \, dx,
\]
\[
= \lim_{b \to 1^-} \int_0^b \frac{1}{(1-x)^2} \, dx + \lim_{a \to 1^+} \int_a^2 \frac{1}{(1-x)^2} \, dx.
\]
To evaluate the integrals we let \( u = 1 - x \) so that \( du = -dx \). Therefore, for the first limit we have
\[
\lim_{b \to 1^-} \int_0^b \frac{1}{(1-x)^2} \, dx = \lim_{b \to 1^-} \int_1^{1-b} \frac{-1}{u^2} \, du = \lim_{b \to 1^-} \left[ \frac{1}{u} \right]_1^{1-b} = \lim_{b \to 1^-} \frac{1}{1-b} - 1 = \infty.
\]
Since the first integral diverges, the integral \( \int_0^2 \frac{1}{(1-x)^2} \, dx \) also diverges.

7. (a) We rewrite the series as
\[
\sum_{n=0}^{\infty} \ln \left( \frac{2+n}{1+n} \right) = \sum_{n=0}^{\infty} -\ln(1+n) + \ln(2+n).
\]
This is a telescoping series whose \( n \)th partial sum is given by:
\[
s_n = -\ln 1 + \ln 2 - \ln 2 + \ln 3 - \ln 3 + \ln 4 + \ldots - \ln n + \ln(n+1),
\]
\[
= -\ln 1 + \ln(n+1) = \ln(n+1).
\]
Therefore, the sum of the series is \( \lim_{n \to \infty} s_n = \lim_{n \to \infty} \ln(n+1) = \infty \) and the series diverges.

(b) Since \( \frac{\ln n}{n^3} < \frac{n}{n^3} = \frac{1}{n^2} \) for \( n \geq 1 \) and \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) converges (\( p \)-series with \( p > 1 \)), the series converges by the Direct Comparison Test.

8. (a) The series is \( \sum_{n=0}^{\infty} \frac{(x-3)^n}{n^3} = \frac{1}{3} \sum_{n=0}^{\infty} \left( \frac{x-3}{4} \right)^n \) which is geometric. It therefore converges absolutely for \( \left| \frac{x-3}{4} \right| < 1 \), which is the interval \((-1, 7)\), and diverges at all other points.

(b) Applying the ratio test to determine absolute convergence we have
\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{|3x-4|^{n+1}}{\sqrt{n+1}} \frac{\sqrt{n}}{|3x-4|^n} = \lim_{n \to \infty} \sqrt{n} |3x-4| = |3x-4|
\]
from which we know that the series converges absolutely on \( |3x-4| < 1 \), which is the interval \((1, 5/3)\), and diverges on \(|3x-4| > 1\).
At the endpoint $x = 5/3$ we have $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$, which is a divergent $p$-series since it has $p = 1/2 \leq 1$.

At the endpoint $x = 1$ we have $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ which is an alternating series. Since the size of the terms is decreasing and goes to zero as $n \to \infty$, the series is convergent by the alternating series test. It is not absolutely convergent because when we take absolute values we again get the divergent $p$-series $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$. Therefore it is conditionally convergent.

The conclusion is that the power series is absolutely convergent for $x$ in $(1, 5/3)$, conditionally convergent at $x = 1$, and divergent at all other points.

9. (a) We use the binomial series $(1 + x)^m = \sum_{k=0}^{\infty} \binom{m}{k} x^k$ with $m = -1/2$, and replace $x$ with $-x^2$. The result is

$$
(1 - x^2)^{-1/2} = \sum_{k=0}^{\infty} \binom{-1/2}{k} (-x^2)^k = \sum_{k=0}^{\infty} \left(\frac{-1/2}{k} \right) (-1)^k x^{2k}
$$

(b) The derivative of $\sin^{-1} x$ is $\frac{1}{\sqrt{1-x^2}} = (1 - x^2)^{-1/2}$, so we can simply integrate the series from part (a):

$$
\sin^{-1} x = \int (1 - x^2)^{-1/2} \, dx = \int \sum_{k=0}^{\infty} \binom{-1/2}{k} (-1)^k x^{2k} \, dx = C + \sum_{k=0}^{\infty} \binom{-1/2}{k} \frac{(-1)^k x^{2k+1}}{2k+1}
$$

The constant $C$ is found to be zero since $\sin^{-1} 0 = 0$. The problem asks for the Taylor polynomial of order 3, so in this case we only need the terms for $k = 0$ and $k = 1$. The coefficients are $\binom{-1/2}{0} = 1$ when $k = 0$ and $-\binom{-1/2}{1/3} = 1/6$ when $k = 1$, so our order 3 Taylor polynomial is $x + \frac{x^3}{6}$.

10. We take $y = \sum_{n=0}^{\infty} a_n x^n$. Then $y' = \sum_{n=1}^{\infty} na_n x^{n-1}$ and the differential equation is

$$
2x^2 = y' - x^2 y = \sum_{n=1}^{\infty} na_n x^{n-1} - x^2 \sum_{n=0}^{\infty} a_n x^n
$$

$$
= a_1 + 2a_2 x + \sum_{m=2}^{\infty} [(m+1)a_{m+1} - a_{m-2}] x^m
$$

from which we see that $a_1 = 0$, $a_2 = 0$, $3a_3 - a_0 = 2$ and that $(m+1)a_{m+1} - a_{m-2} = 0$ for $m \geq 3$. We can put this last equation into the more useful form $a_{m+1} = a_{m-2}/(m+1)$, or the even more convenient $a_{n+3} = a_n/(n+3)$.

Using the initial condition we have $a_0 = y(0) = -1$, so we can now work out all of the coefficients from the above equations.

$$
a_0 = -1 \implies a_3 = \frac{1}{3} \implies a_6 = \frac{1}{6} \cdot \frac{1}{3} \implies a_9 = \frac{1}{9} \cdot \frac{1}{6} \cdot \frac{1}{3}$$

$$
a_1 = 0 \implies a_4 = 0 \implies a_7 = 0 \implies a_{10} = 0$$

$$
a_2 = 0 \implies a_5 = 0 \implies a_8 = 0 \implies a_{11} = 0$$

This then gives the first four non-zero terms of the series to be

$$
y = -1 + \frac{1}{3} x^3 + \frac{1}{18} x^6 + \frac{1}{162} x^9 + \cdots
$$