1. (a) Write down the general form of the partial fraction expansion of

\[ \frac{x^3 - 9x + 4}{(x - 1)^3(x + 2)(x^2 + 1)^2} \]

DO NOT ATTEMPT TO EVALUATE THE VALUES OF THE COEFFICIENTS.

(b) Evaluate \( \int \frac{x^2 + x}{(x^2 + 1)(x - 1)} \, dx \).

Solution

(a) The factor \( x^2 + 1 \) is an irreducible quadratic, so the denominator is fully factored and the general form is

\[ \frac{x^3 - 9x + 4}{(x - 1)^3(x + 2)(x^2 + 1)^2} = \frac{A}{(x - 1)^3} + \frac{B}{(x - 1)^2} + \frac{C}{x - 1} + \frac{D}{(x + 2)} + \frac{Ex + F}{(x^2 + 1)^2} + \frac{Gx + H}{x^2 + 1} \]

(b) We have the partial fraction decomposition

\[ \frac{x^2 + x}{(x^2 + 1)(x - 1)} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x - 1} = \frac{Ax^2 - Ax + Bx - B + Cx^2 + C}{(x^2 + 1)(x - 1)} \]

so that we get the simultaneous equations \( A + C = 1, -A + B = 1 \) and \(-B + C = 0\). Then \( B = C = 1 \) and \( A = 0 \), so

\[ \int \frac{x^2 + x}{(x^2 + 1)(x - 1)} \, dx = \int \frac{1}{x^2 + 1} \, dx + \frac{1}{x - 1} \, dx = \tan^{-1} x + \ln |x - 1| + C \]

2. Evaluate \( \int \frac{2}{(1 + x^2)^2} \, dx \)

Solution

Let \( x = \tan \theta \). Then \( dx = \sec^2 \theta \, d\theta \) and

\[ (1 + x^2) = (1 + \tan^2 \theta) = (\sec^2 \theta)^2 = \sec^4 \theta \]

so we have

\[ \int \frac{2}{(1 + x^2)^2} \, dx = \int \frac{2 \sec^2 \theta}{\sec^4 \theta} \, d\theta = \int \frac{2 d\theta}{\sec^2 \theta} = \int 2 \cos^2 \theta \, d\theta = \int (\cos 2\theta + 1) \, d\theta = \frac{\sin 2\theta}{2} + \theta + C \]

but \( \sin 2\theta = 2 \sin \theta \cos \theta \) and \( x = \tan \theta \), so we have (by drawing the appropriate triangle) \( \sin \theta = x/\sqrt{1 + x^2} \) and \( \cos \theta = 1/\sqrt{1 + x^2} \). Therefore

\[ \int \frac{2}{(1 + x^2)^2} \, dx = \frac{x}{(1 + x^2)} + \tan^{-1} x + C \]
3. Are the following convergent? Justify your answer.

(a) \[ \int_{1}^{\infty} e^{-x} \, dx \]
(b) \[ \int_{1}^{\infty} e^{-x^2} \, dx \]
(c) \[ \int_{0}^{\infty} e^{-x^2} \, dx \]

Solution

(a) This is an improper integral because we are integrating over an infinite interval. However we can directly compute

\[
\int_{1}^{\infty} e^{-x} \, dx = \lim_{b \to \infty} \int_{1}^{b} e^{-x} \, dx = \lim_{b \to \infty} [ -e^{-x} ]_{1}^{b} = \lim_{b \to \infty} ( -e^{-b} + e^{-1} ) = e^{-1}
\]

so this integral is convergent.

(b) Notice that \( x^2 \geq x \) on \([1, \infty)\) so \( e^{-x^2} \leq e^{-x} \). As both of these are positive functions we can use the Direct Comparison Test and the result of Part (i) above to see that \( \int_{1}^{\infty} e^{-x^2} \, dx \) is convergent.

(c) Since \( \int_{0}^{1} e^{-x^2} \, dx \) is the integral of a continuous function on a finite interval it is not improper, and we can just write

\[
\int_{0}^{\infty} e^{-x^2} \, dx = \int_{0}^{1} e^{-x^2} \, dx + \int_{1}^{\infty} e^{-x^2} \, dx
\]

and using the result of part (b) we see that the integral is convergent.

4. (a) Show that the sequence \( s_n = \ln(\sqrt{2n}) \) converges to 0.

(b) Does the sequence \( b_n = (\sin n) \ln(\sqrt{2n}) \) converge? Justify your answer.

(c) You are given a series with \( \sum_{j=1}^{n} a_j = s_n \), where \( s_n \) is as in part (a) above. What is \( \sum_{j=1}^{\infty} a_j \)?

Solution

(a) Let \( f(x) = \ln(\sqrt{2x}) \) for \( x > 0 \). Then \( s_n = f(n) \). We can apply L'Hôpital's rule to the function \( f(x) \) to compute the limit as \( x \to \infty \) as follows

\[
\lim_{x \to \infty} \ln(\sqrt{2x}) = \lim_{x \to \infty} \ln((2x)^{1/2}) = \lim_{x \to \infty} \frac{\ln 2x}{x} = \lim_{x \to \infty} \frac{2/2x}{1} = \lim_{x \to \infty} \frac{1}{x} = 0
\]

and we have a theorem from class that implies the limit of \( s_n \) as \( n \to \infty \) is equal to the limit of \( f(x) \) as \( x \to \infty \).

(b) We use the squeeze theorem. Since \( |\sin n| \leq 1 \) we have

\[
0 \leq |b_n| = |\sin n| \ln(\sqrt{2n}) \leq \ln(\sqrt{2n}) = s_n
\]

and the squeeze theorem and the result of part (a) implies that \( b_n \to 0 \).
(c) We are given that the number $s_n$ is the $n$-th partial sum of the series $\sum_{j=1}^{\infty} a_j$. By definition, the sum of the series is the limit of the partial sums, so using part (a)

$$\sum_{j=1}^{\infty} a_j = \lim_{n \to \infty} s_n = 0$$

5. Is $\sum_{n=1}^{\infty} \frac{1}{n(1 + \ln n)^2}$ convergent? Justify your answer.

**Solution**

We apply the integral test. The function $f(x) = \frac{1}{x(1 + \ln x)^2}$ is continuous, positive and decreasing on $[1, \infty)$ and has $f(n)$ equal to the $n$-th term of the series, so we compute the improper integral $\int_{1}^{\infty} f(x) \, dx$. We do this using the substitution $u = 1 + \ln x$, for which $du = \frac{dx}{x}$. Notice that the upper limit of integration becomes $1 + \ln b$ in the substitution and the lower limit is $1 + \ln 1 = 1$.

$$\int_{1}^{\infty} \frac{dx}{x(1 + \ln x)^2} = \lim_{b \to \infty} \int_{1}^{b} \frac{dx}{x(1 + \ln x)^2} = \lim_{b \to \infty} \int_{1}^{1 + \ln b} \frac{du}{u^2} = \lim_{b \to \infty} \left[ -u^{-1} \right]_{1}^{1 + \ln b} = \lim_{b \to \infty} -(1 + \ln b)^{-1} + 1 = 1$$

where we used the fact that $\ln b \to \infty$ when $b \to \infty$. Having shown that the integral converges, we know from the integral test that the series also converges.

6. (a) Show that $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ is convergent.

(b) For which $x > 0$ is $\sum_{n=1}^{\infty} \frac{n^x}{x^n}$ convergent? Justify your answer.

**Solution**

(a) We use the Ratio Test and compute the limit as $n \to \infty$ of the $(n+1)$th term divided by the $n$-th term:

$$\lim_{n \to \infty} \frac{(n+1)^2/2^{n+1}}{n^2/2^n} = \lim_{n \to \infty} \frac{(n+1)^2}{n^2} \frac{2^n}{2^{n+1}} = \lim_{n \to \infty} \frac{n+1}{n} \left( \frac{1}{2} \right)^2 = \frac{1}{2} \left( \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right) \right)^2 = \frac{1}{2} \cdot 1^2 = \frac{1}{2}$$

As this has absolute value less than 1 the series converges.

(b) We use the same method as in the first part, keeping in mind that $x$ is constant here - the limit is only with respect to $n$.

$$\lim_{n \to \infty} \left( \frac{n+1}{n} \right)^x \frac{x^n}{x^{n+1}} = \frac{1}{x} \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^x = \frac{1}{x} \cdot 1^x = \frac{1}{x}$$

Now the ratio test says that this converges when this limit has absolute value less than 1, which happens when $\frac{1}{|x|} < 1$, so when $|x| > 1$, but we are given $x > 0$, so this becomes just the interval $x > 1$. The ratio test also tells us that the series diverges when $|x| < 1$, so for $x > 0$ this is the interval $0 < x < 1$. All that remains is to deal with the case $x = 1$, where the series is $\sum_{n=1}^{\infty} n$, which diverges.

Therefore the $x > 0$ for which the series is convergent are those for which $x > 1$. 