Math 671
Fall 2003, Final Problem Set
Due: Friday, December 12, 5:00 pm

- You are not to discuss these problems with anyone or show this examination to anyone before turning in the exam.
- You may use your book, notes, and any other books. You can see me or e-mail me (lawler@math.cornell.edu). If the error/clarification is significant, I will post a correction on the web page of the course and (if found soon enough) announce the change in class.

1. Let $X_1, X_2, \ldots$ be i.i.d random variables with $\mathbf{E}[X_1] = 0$, $\mathbf{P}(X_i > 0) > 0$, $\mathbf{P}(|X| > 3) = 0$. Let $\theta \in \mathbb{R} \setminus \{0\}$, and let

$$M_0 = 1, \quad M_n = m(\theta)^{-n} e^{\theta(X_1 + \cdots + X_n)},$$

where $m(\theta) = \mathbf{E}[e^{\theta X_1}]$ is the moment generating function for $X_1$. Show that $M_n$ is a martingale and that there exists an $M_\infty$ such that $M_n \to M_\infty$ w.p.1. Is $M_n$ a uniformly integrable martingale?

2. True or false: Suppose $X_1, X_2, \ldots$ are independent, mean zero (not identically distributed) random variables. Suppose that it is known that w.p.1,

$$\lim_{n \to \infty} \frac{X_1 + \cdots + X_n}{n} = c,$$

for some $c \in \mathbb{R}$. Then $c$ must equal zero.

3. Let $B_t^1, B_t^2$ denote two independent standard Brownian motions defined on the same probability space. Let $B_t = B_t^1 + B_t^2$. Show that $B_t$ is a Brownian motion with drift 0 and variance $\sigma^2$. What is $\sigma^2$?

4. Let $B_t$ be a standard Brownian motion. Does there exist a stopping time $T$ such that: $\mathbf{P}(T < \infty) = 1$, $\mathbf{P}(B_T \leq 0) = 0$; $\mathbf{E}[B_T] = \infty$? Justify your answer.

5. Let $X_1, X_2, \ldots$ be independent, identically distributed random variables with $\mathbf{P}(X_j \neq 0) > 0$. Consider the random Taylor series

$$f(z) = \sum_{n=0}^{\infty} X_n z^n.$$ 

Let $R$ denote the radius of convergence of this power series. Show that $R$ is nonrandom and its value is contained in the set $\{0, 1\}$. Show that $R = 0$ if and only if $\mathbf{E}[\log^+ |X_1|] = \infty$. (Recall that $\log^+ x = \max\{\log x, 0\}$.)

6. Let $B_t$ be a standard Brownian motion and $T$ an independent exponential random variable defined on the same probability space with $\mathbf{E}[T] = 1$. Find the following quantities

$$\mathbf{E}[B_T^2],$$

$$\mathbf{P}(B_t > 1 \text{ for some } 0 \leq t \leq T),$$
7. True or false: Suppose $X_1, X_2, \ldots$ are independent random variables $\mathbb{P}\{X_j = 1\} = \mathbb{P}\{X_j = -1\} = 1/2$. Then with probability one the limit
\[
\lim_{n \to \infty} \sum_{j=2}^{n} \frac{X_j}{\log j^j}
\]
exists.

8. True or false: Let $X_1, X_2, \ldots$ be as in Problem 7. Let
\[
S_n = \sum_{j=1}^{n} j X_j,
\]
Then there exists a sequence of positive numbers $a_n$ such that $a_n^{-1} S_n$ converges in distribution to a $N(0, 1)$ random variable.

9. Find
\[
\lim_{n \to \infty} e^{-n^2} \sum_{j=0}^{n^2+n} \frac{n^{2j}}{j!}.
\]
(Hint: consider Poisson distributions)