5 Counting Real Numbers

5.1 Decimals

Every real number \( r \) has a multiplicity of Cauchy sequences representing it. The most familiar representative is known as a decimal expansion of \( r \). We begin this section by describing how to obtain this from the perspective we have been developing.

First, let \( \{a_n\} \) be any sequence of natural numbers between 0 and 9, inclusive. Define a sequence \( \{b_n\} \) by the following rule:

\[
b_n = a_1 10^{-1} + a_2 10^{-2} + \ldots + a_n 10^{-n}.
\] (1)

Any sequence \( \{b_n\} \) given by such an equation, where the \( a_i \) are integers between 0 and 9 is called a decimal expansion, no matter how the \( a_i \) happen to be selected.
**Exercise 1.** Check that any decimal expansion is a Cauchy sequence of rational numbers.

If \( \{b_n\} \) is a decimal expansion, let \( s \) be the real number it represents. Then we call \( \{b_n\} \) a *decimal representation* of \( s \).

**Exercise 2.** Check that \( s \) is a real number in the closed interval \([0, 1]\).

Of course, any decimal expansion \( \{b_n\} \) as above is usually written more familiarly as \( 0.a_1a_2\ldots a_n\ldots \).

We now show that any real number \( r \) belonging to the half-open interval \([0, 1)\) has at least one decimal expansion.

Subdivide the interval \([0, 1)\) into ten equal parts, \([0, 1/10)\), \([1/10, 2/10)\), \ldots, \([9/10, 1)\), which we label \( 0, \ldots, 9 \), from left to right. The given real number \( r \) belongs to exactly one of these subintervals. Let \( a_1 \) be the number between 0 and 9 that labels this subinterval. Then, take that subinterval, subdivide it into ten equal parts similarly, labeling the parts according to the same scheme, and let \( a_2 \) be the number labeling the unique sub-subinterval in which \( r \) lies. Continue similarly inductively. This produces the sequence \( \{a_n\} \) and, using equation (1), we get the decimal expansion \( \{b_n\} = .a_1a_2a_3\ldots \). Let \( s \) be the real number represented by the constructed decimal expansion \( .a_1a_2a_3\ldots \). Notice that, by the construction, \( b_n \) is the left-hand endpoint of the subinterval containing the given real number \( r \) at step \( n \) of the construction. So, by construction, \( |r - b_n| < 10^{-n} \). It follows easily from this that \( r = s \), i.e., that \( r \) is represented by the decimal \( .a_1a_2a_3\ldots \).

**Exercise 3.** Verify that \( r = s \).

If we are interested in a general real number \( r \), we note that it belongs to a half-open interval \([N, N + 1)\), for some unique natural number \( N \). We then apply the
foregoing construction to the real number \( r - N \), obtaining a decimal expansion for it. We can also write \( N \) as a finite decimal in the usual way. We then place this last to the left of the decimal point, preceding the expansion just obtained for \( r - N \). This gives the usual “two-sided” decimal expansion for \( r \). Since the integer part of the decimal, that is, \( N \), is not particularly interesting for our purposes, we often restrict to real numbers in \([0, 1)\), but not always.

This argument shows that every real number has at least one decimal representation, but it does not claim that such a representation is unique. In fact, there are well-known examples of real numbers that have more than one decimal representation. For example, the integer 1 can be represented as 1.000\ldots and also as .9999\ldots. From our perspective, this is not particularly mysterious. It’s simply a special reflection of the general fact that two Cauchy sequences of rational numbers may converge to the same real number. Or, in our earlier terminology, two Cauchy sequences of real numbers may differ by a Cauchy sequence in \( \mathbb{Z} \). The following theorem states that the only cases in which this happens for decimal expansions are when there is an infinite string of 9’s and an infinite string of 0’s as in the given example. In the theorem, we shall restrict to real numbers in the interval \([0, 1)\), since the integer part of a real number is always uniquely determined.

**Exercise 4.** Verify that if \( r \) is a real number, and \( M \) and \( N \) are integers such that \( r \in [M, M + 1) \cap [N, N + 1) \), then \( M = N \).

**Theorem 1 (Failure of uniqueness in decimal expansion).** Suppose that \( r \) is a real number in the half-open interval \([0, 1)\) with decimal expansions \(.b_1b_2b_3\ldots\) and \(.c_1c_2c_3\ldots\), and suppose that, for some natural number \( i \), \( b_i > c_i \). Then, there is a unique natural number \( m \) such that: (a) \( b_j = c_j \), when \( j < m \); (b) \( b_m = c_m + 1 \); (c)
$b_j = 0$ and $c_j = 9$, when $j > m$.

Notice, in particular, that (c) implies that the real number $r$ is a rational number.

More informally, the theorem says that if a real number $r$ has two distinct decimal expansions, then the number must be a rational number, and the two expansions are identical up to, but not including, a certain term, say the $m^{th}$; the $m^{th}$ term of one is exactly 1 bigger than the $m^{th}$ term of the other; and this bigger term is followed exclusively by 0’s while the other is followed exclusively by 9’s.

Thus, a decimal representation of every real number is almost unique: the only exceptions are rational numbers of the type described in the theorem, for which there are precisely two representations. The proof of this theorem is not very difficult, requiring only careful attention to the properties of decimal expansions. The proof will be left to the intrepid reader.

If we wish to be sure that the decimal expansions are unique, we can accomplish this by calling a decimal expansion admissible if it does not contain infinitely many consecutive 9’s. Then every real number in $[0, 1)$ has a unique admissible decimal expansion.

The decimal expansion of a real number is so convenient and natural—after all it’s based on counting with fingers and toes—that it is often taken as the definition of a real number. From our perspective, however, this form of representation is just one of many possible ones, some of which are more convenient in other contexts (e.g., binary representation for computer codes). We shall have occasion to use the decimal representation in our discussion of the Cantor diagonal method; in fact, we shall also make use of the non-uniqueness theorem just asserted.
5.2 Cardinalities of some infinite subsets of \( \mathbb{R} \)

The student is referred to earlier sections in both the *Set Theory* and *Natural Numbers* notes where we introduce the basic concepts involving infinite and finite cardinalities.

Although we did not point this out explicitly in Exercise 23 of the *Natural Numbers* notes, that exercise implies that every infinite set \( S \) contains a subset that is equipotent to \( \mathbb{N} \). It follows that \( \text{card}(\mathbb{N}) \leq \text{card}(S) \) for any infinite \( S \), which is another way of saying that \( \text{card}(\mathbb{N}) \) is the smallest infinite cardinality.

**Definition 1.** We call a set \( S \) **countable** provided \( \text{card}(S) \leq \text{card}(\mathbb{N}) \). Some mathematicians use the term **denumerable** to mean the same thing. To emphasize that a countable set is not finite, we may call it **countably infinite**.

A set that is not countable is called **uncountable**.

By the transitivity of the order relation for cardinalities, any subset of a countable set is countable. Or, to assert the contrapositive: any set that contains an uncountable set is uncountable.

We point out right away that there are many sets known to be uncountable. Indeed, given any infinite set \( S \), the power set \( 2^S \) is uncountable: this is an immediate consequence of the extra-credit exercise for Week 6. However, sets like these power sets are not in common use in introductory mathematics. What interests us here is whether some of the sets that are in common use, say in calculus courses, are countable or uncountable.

**5.2.1 Some standard countable sets.**

**Exercise 5.** Define the function \( f : \mathbb{Z} \to \mathbb{N} \) by the rule
\[ f(n) = \begin{cases} 
2n, & \text{if } n \geq 0, \\
-2n + 1, & \text{if } n < 0. 
\end{cases} \]

Prove that \( f \) is a bijection.

This proves:

**Proposition 1.** \( \mathbb{Z} \) is countable.

Now, let \( d \) be any positive integer, and consider the set \( \mathbb{N}^d \) of all \( d \)-tuples of natural numbers. Choose any \( d \) distinct prime numbers \( p_1, p_2, \ldots, p_d \). Such primes always exist because there are infinitely many primes.

**Exercise 6.** Define the function \( g : \mathbb{N}^d \to \mathbb{N} \) by the rule

\[ g(n_1, n_2, \ldots, n_d) = p_1^{n_1} p_2^{n_2} \cdots p_d^{n_d}. \]

Prove that \( g \) is injective. (You may use, without proof the uniqueness of prime factorization.)

This exercise shows that \( card(\mathbb{N}^d) \leq card(\mathbb{N}) \), and so we may conclude:

**Proposition 2.** \( \mathbb{N}^d \) is countable.

Here is a simple consequence of this proposition when \( d = 2 \)

**Corollary 3.** Suppose that \( \{S_i\} \) is a collection of countable sets, with \( i \) ranging over \( \mathbb{N} \). Then, the union \( \bigcup_i S_i \) is countable.

**Proof.** For each \( i \) there is an injection \( h_i : S_i \to \mathbb{N} \). (If \( S_i \) is infinite, we can choose \( h_i \) to be a bijection, but if \( S_i \) is finite, we can only choose \( h_i \) to be an injection. We
only need it to be an injection.) We define a function \( f : \bigcup S_i \to \mathbb{N}^2 \) as follows: Choose any element \( x \in \bigcup S_i \), and let \( i(x) \) be the smallest natural number \( i \) such that \( x \in S_i \). Such numbers \( i \) must exist, by definition of the union \( \bigcup S_i \). For each \( x \), the number \( i(x) \) is unique, because it is the smallest \( i \) with \( x \in S_i \). Now, define \( f(x) = (i(x), h_i(x)) \in \mathbb{N} \times \mathbb{N} = \mathbb{N}^2 \). If \( f(y) = f(x) \), then \( i(y) = i(x) \) and \( h_i(y)(y) = h_i(x)(x) \). Set \( j = i(x) = i(y) \). Then, the preceding equality can be written as \( h_j(y) = h_j(x) \). Since we know that \( h_j \) is injective, it follows that \( y = x \). Therefore, \( f \) is injective, implying that \( \text{card}(\bigcup S_i) \leq \text{card}(\mathbb{N}^2) \leq \text{card}(\mathbb{N}) \), as desired.

Notice if even one of the sets \( S_i \) above is countably infinite, then the union is countably infinite. The union is finite, precisely when each of the sets \( S_i \) is finite and all of them are contained in a union of a finite number of them. The case that interests us the most is the case when all are infinite.

Now suppose that, for each number \( i = 1, 2, \ldots, d \), \( S_i \) is a countable set. For each \( i \), choose a bijection \( h_i : S_i \to \mathbb{N} \). Then, define \( H : S_1 \times S_2 \times \ldots \times S_d \to \mathbb{N}^d \) by the rule \( H(s_1, s_2, \ldots, s_d) = (h_1(s_1), h_2(s_2), \ldots, h_d(s_d)) \). (Recall that \( S_1 \times S_2 \times \ldots \times S_d \) is the Cartesian product of the sets \( S_1, S_2, \ldots, S_d \), as described in the Set Theory notes. It consists of all \( d \)-tuples \( (s_1, s_2, \ldots, s_d) \), where each \( s_i \) ranges over the set \( S_i \).)

**Exercise 7.** Prove that \( H \) is a bijection.

Therefore, the set \( S_1 \times S_2 \times \ldots \times S_d \) is countable. If all the \( S_i \) are equal to the same set \( S \), then the Cartesian product of all of them can be written as a “power” \( S^d \), just as we did for \( \mathbb{N}^d \) above.

Since \( \mathbb{Z} \) is countable, we can now conclude

**Corollary 4.** \( \mathbb{Z}^d \) is countable.
Exercise 8. Let \( r \) be any rational number.

a. Prove that there exists a unique ordered pair of integers \((a, b)\) such that (i) \(a > 0\), ii) \( r = b/a \), and (iii) the only positive integer dividing both \( a \) and \( b \) is 1.
   (You may use your knowledge of integer division for this.)

b. Define the function \( k : \mathbb{Q} \rightarrow \mathbb{Z}^2 \) by the rule \( k(r) = (a, b) \), where \((a, b)\) is as described in a). Prove that \( k \) is injective.

It is now almost immediate to conclude the following

Theorem 2. \( \mathbb{Q} \) is countably infinite.

We let the reader fill in the remaining couple of details.

Let \( \mathbb{A} \) denote the set of all real numbers that are roots of non-zero polynomials (in one variable) with rational coefficients. These numbers are known as the real algebraic numbers. If \( r \in \mathbb{Q} \), then \( r \) is a root of the polynomial (with rational coefficients) \( P(x) = x - r \). So, \( r \) is an algebraic number. Therefore, \( \mathbb{Q} \subseteq \mathbb{A} \). However, set equality does not hold because there are algebraic numbers, such as \( \sqrt{2} \), that are not rational. Nevertheless, we can prove the following:

Theorem 3. \( \mathbb{A} \) is countably infinite.

The proof follows from some of the following exercises (cf. Exercise 11).

It follows that \( \text{card}(\mathbb{Q}) = \text{card}(\mathbb{A}) \), even though \( \mathbb{A} \) is a set that contains not only \( \mathbb{Q} \), but also an infinite number of irrational numbers as well.

In fact, for those students who are familiar with the concept of a field, it may not come as a surprise that \( \mathbb{A} \) is a field. This fact is less obvious than it seems, but you needn’t worry. We won’t be going into details or using this fact here.
Recall that the degree of a polynomial is the highest power of $x$ appearing in the polynomial with a non-zero coefficient.

**Exercise 9.** Prove that there is a bijection between the set of all degree-$d$ polynomials with rational coefficients and the set $\mathbb{Q}^{d+1}$.

It follows from this exercise and results above that the set of all polynomials with rational coefficients having degree $d$ is countable. Now, each such polynomial is known to have at most $d$ distinct real roots.

**Exercise 10.** Prove that the set of all real numbers that are roots of degree-$d$ polynomials with rational coefficients is countable.

**Exercise 11.** Use the Exercise 10, together with Corollary 3 to prove Theorem 3.

### 5.2.2 Some uncountable subsets of $\mathbb{R}$.

The most notable subset of $\mathbb{R}$ is, of course $\mathbb{R}$ itself, so we announce its uncountability first.

**Theorem 4.** $\mathbb{R}$ is uncountable.

A proof will be given below.

Since $\mathbb{Q}$ and $\mathbb{A}$ are countable subsets of $\mathbb{R}$, it follows that $\text{card}(\mathbb{Q}) < \text{card}(\mathbb{R})$ and $\text{card}(\mathbb{A}) < \text{card}(\mathbb{R})$.

We wish to describe another important uncountable set. Let us denote the set of all real numbers that are not algebraic by $\mathbb{T}$. This is the set of real transcendental numbers. Two examples of such numbers are the well-known quantities $e$ and $\pi$. The proofs that these numbers are not algebraic are quite difficult, particularly in
the case of \( \pi \). The number \( e \) was first proved to be transcendental in 1873 by the French mathematician Charles Hermite; nine years later, the German mathematician Ferdinand von Lindemann proved that \( \pi \) is transcendental.

**Theorem 5.** \( \mathbb{I} \) is uncountable. In fact, \( \text{card}(\mathbb{I}) = \text{card}(\mathbb{R}) \).

We now prove a result slightly stronger than Theorem 4, which implies that theorem as well as the first half of Theorem 5. We’ll call a subset \( S \) of the reals exhaustive if \( S = \mathbb{R} \).

**Proposition 5.** No countable subset \( S \) of \( \mathbb{R} \) is exhaustive.

**Proof.** Suppose that \( S \) is a given countable subset of \( \mathbb{R} \). Of course, if \( S \) is finite, it cannot be exhaustive, so we may as well assume that \( S \) is countably infinite. Then, we can write \( S \) as \( S = \{r_0, r_1, r_2, \ldots \} \), where the \( r_i \)'s denote distinct real numbers, one for each \( i \). As discussed in the previous section, we can write each \( r_i \) uniquely as a decimal, as follows: \( r_i = N_i.a_{1i}a_{2i}a_{3i} \ldots \), for each natural number \( i \). Here \( N_i \) is the unique greatest integer not exceeding \( r_i \), and the terms to the right of the decimal points comprise the unique, admissible decimal expansion of \( r_i - N_i \).

To prove the proposition, we must now construct a real number \( s \) that is not in this set. We construct \( s \) as follows: We write \( s = N.c_1c_2c_3 \ldots \) and then define \( N \) and the terms \( c_i \). We set \( N = N_0 + 1 \). For each \( i \geq 1 \), we define \( c_i = 1 \) if \( a_{ii} \neq 1 \) and \( c_i = 2 \) if \( a_{ii} = 1 \). Notice that \( c_i \neq a_{ii} \), for every \( i \). The integer \( N \) and the terms \( c_i \) are all well-defined, and there are no 9’s in the decimal. So, we have defined a real number \( s \), with \( N \) the greatest integer not exceeding \( s \) and \( s - N \) having \( .c_1c_2c_3 \ldots \) as its admissible decimal expansion. We now show that \( s \) is not in the set \( S \).

If \( s \in S \), then \( s \) must equal one of the real numbers \( r_i \), say \( s = r_k \). So, by the uniqueness results established before, the integer part of \( s \) and that of \( r_k \) must be the
same, and every term of their decimal expansions must be the same. The question is: Which \( r_k \) can \( s \) be? It certainly cannot equal \( r_0 \), because \( N = N_0 + 1 \neq N_0 \). Suppose \( s = r_k \), for some \( k > 0 \). Then \( c_k \) would equal \( a_{kk} \), contradicting the definition of \( c_k \). So that’s not possible. Therefore, \( s \) does not appear in the set \( S \).

Since no countable subset of \( \mathbb{R} \) is exhaustive, \( \mathbb{R} \) cannot be countable, since, by definition \( \mathbb{R} \) is exhaustive. This proves Theorem 4.

But, we can show more than this. Suppose that \( T \) is countable. Then so is \( A \cup T = \mathbb{R} \), a contradiction. Therefore, \( T \) is also uncountable. This proves the first part of Theorem 5.

The second part of Theorem 5 requires a bit more work. It follows immediately from the following theorem.

**Theorem 6.** Let \( X \) be an uncountable set, and let \( Y \) be a countable subset of \( X \). Then the complement \( X \setminus Y \) has the same cardinality as \( X \).

**Proof.** Since \( Y \) is countable, we can write it as \( \{y_0, y_1, y_2, \ldots\} \). The complement \( X \setminus Y \) must be infinite, for if not, the union \( (X \setminus Y) \cup Y \), which equals \( X \), would be countable, by Corollary 3. Since \( X \setminus Y \) is infinite, it contains a countable subset, say, \( W = \{w_0, w_1, w_2, \ldots\} \). We now define a function \( h : X \to X \setminus Y \). Choose any \( x \in X \). If \( x = w_i \), let \( h(x) = w_{2i} \). If \( x = y_i \), let \( h(x) = w_{2i+1} \). So far, \( h \) maps \( W \cup Y \) bijectively to \( W \). Finally, if \( x \) is not in \( W \cup Y \), then set \( h(x) = x \). We let the reader verify that \( R(h) \subseteq X \setminus Y \) and that \( h : X \to X \setminus Y \) is a bijection.

This implies the second part of Theorem 5 by taking \( X = \mathbb{R} \) and \( Y = A \).
5.2.3 Concluding comments

We conclude by noting how remarkable all of these theorems are. First, we know that the natural numbers form just a small subset of the rational numbers, yet we are asserting that the two sets are equipotent! Even more remarkable, the same holds for the set of real algebraic numbers. But most remarkable of all, the set of all real numbers is immeasurably larger than either of these. The proof that this is the case was discovered by Georg Cantor, a German-Jewish mathematician of the 19th century who was responsible for much of the early development of set theory. The proof, which we gave above in proving Proposition 5 has become known as Cantor’s diagonal process—this is because the proof involves looking at the decimal terms $a_{ii}$; if we write down all the decimal expansions in the proof, each below its predecessor, the $a_{ii}$’s line up on a diagonal. This diagonal process has played an important role in the development of mathematics in the 20th century. For example, a version of the diagonal process is one of the two ingredients at the heart of the proof of the renowned Incompleteness Theorem of Kurt Gödel.

Similarly, Theorem 5 asserts that the cardinality of the set of real transcendental numbers is greater than that of the real algebraic numbers. However, although algebraic numbers comprise only 0% of all the reals\footnote{Technically, what we are saying here is that there is zero probability that a real number chosen randomly will be algebraic}, much more is known about them than about transcendental numbers. Indeed, mathematicians are hard pressed to write down a significant number of transcendental numbers. In 1844, the French mathematician Joseph Liouville showed how to construct a family of transcendental numbers, the most famous of which has the following decimal expansion: for each $k$, the $k^{th}$ term in the decimal expansion equals 0, except in those cases when $k$ has the
form \( n! \). When \( k \) does have that form, the \( k^{th} \) term is defined to be 1. Essentially, the decimal expansion consists of ever lengthening strings of 0’s separated by ever sparser 1’s. The numbers constructed by Liouville are called (appropriately enough) **Liouville numbers**. However, like algebraic numbers, Liouville numbers can be proved to comprise 0% of the reals. We have already commented on how difficult it was to establish that the familiar real constants \( \pi \) and \( e \) are both transcendental. Since then, there have been other results that show how to produce transcendental numbers. One of the most famous of these is \( 2^{\sqrt{2}} \), which was proved to be transcendental in 1934 by the Russian mathematician Aleksander Gelfond, solving a problem posed around 1900 by David Hilbert. For another example, \( \log_{10}(n) \) was shown to be transcendental, whenever \( n \) is a natural number that is not a power of 10. But there are also many open elementary questions as to whether certain so-called “known” quantities are transcendental, algebraic, or even rational. As far as I know at this writing, these questions are open for the simple-looking quantities \( e + \pi, e\pi, \pi^{\pi}, e^{\pi} \). Needless to say, these and similar questions have given rise to an active area of mathematical research.

There is, however, one transcendental number whose decimal expansion is very easy to write down. This is known as **Mahler’s number**, named after the British mathematician Kurt Mahler, who proved it is transcendental in the 1930’s. To write it down, simply write down all the natural numbers in their usual decimal form, one after another, placing a decimal point after 0: 0.123456789101112131415161718192021 . . . , etc. It seems fitting to end this constructive development of the real numbers with a number that is both mysterious and yet repeats the elementary sequence of natural numbers with which we began.