

Symbolic Logic II: The Predicate Calculus[†]

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1. INTRODUCTION

The predicate calculus is an extension of the propositional calculus that includes the notion of quantification. Instead of dealing only with statements, which have a definite truth-value, we deal with the more general notion of *predicates*, which are assertions in which *variables* appear. The author Solow calls these *conditional statements*. These variables are given as “ranging” over some given sets, and quantification is a process that applies to these variables. Statements are here viewed as special predicates in which there are either no variables at all or in which *all* variables have been quantified. All of the operations of the propositional calculus extend to predicates virtually without change. What needs to be understood, however, is how these operations interact with quantification

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We shall begin by giving a few simple examples of variables and quantification and then turn to describing what we mean by these in general.

2. SOME EXAMPLES

Variables occur in all parts of mathematics, starting with basic algebra, set theory, number theory, and calculus.

For example, the concept of a *polynomial* includes the notion of a variable. When we write $x^3 + 17x + 10$, we think of x as a variable ranging over some set of numbers, say the real numbers. The polynomial itself is a recipe for performing certain algebraic operations on the variable x .

In set theory, we often consider two sets S and T and then, in some manner, define a function $f : S \rightarrow T$. We denote the function values by $f(s)$, where s denotes a variable ranging over S . Sometimes, we may wish to write $t = f(s)$ and call s the independent variable and t the dependent variable.

In calculus, we frequently do what the preceding paragraph just described. The notion of a variable is particularly well-suited to calculus, since it studies the relative rates of different variables, integration with respect to a variable, etc.

In these areas, we often make statements in which we refer to a certain set of values that the variable ranges over. For example, we may say that the polynomial expression $x^2 + 1$ assumes positive values *for all* values of the real variable x . Or we may say that $\cos^2(x) \leq 1$ *for all* real x . Alternatively, we may wish to say that a certain equation has a solution, as when we say “The polynomial equation $x^3 + 17x + 10 = 0$ has a real root.” This may be expressed as “*There exists* an x such that $x^3 + 17x + 10 = 0$,” provided it is understood that x ranges over the real numbers.

These two ways of qualifying the range of values that the variable is permitted to attain—namely, *for all* and *there exists* — are known as *quantification*: the first is so-called *universal* quantification, the second *existential* quantification.

We now make some of these ideas more precise.

3. GENERAL ELEMENTS OF SETS.

We shall assume that students have seen the basic definitions of sets and functions, so we do not go into these in systematic detail here. In a later chapter, we review some ideas from the theory of sets and functions that will be used subsequently.

No matter how a set S is defined or presented, there is a notion of a *general element* of S . This is conceived as an object about which we know nothing other than it belongs to S . It has all of the properties needed to qualify as a member of S but nothing further. Put another way, a general element of S has exactly those properties shared by all elements of S .

For example, if S is the set of all positive reals, then all we know about a general element of S is that it is real and > 0 . So, 3 , $\sqrt{2}$, and π are positive reals, but they are not general elements of S because we know more about them than that they are real and > 0 .

Clearly the notion of a general element of a set S is an abstract concept, since any concrete instance of an element in S has something particular about it which is what enables us to single it out. The whole point of the concept is to enable us to reason conveniently about an element of S *qua element of S*, without inadvertently introducing additional restrictive identifiers.

Another term typically used by mathematicians as a synonym for “general element” is “arbitrary element,” as in the sentence, “Let x be an arbitrary element of S .” It is a common error for students to misunderstand this term and to then select a specific element of S rather than to attend to a general one.

When the author Solow discusses his so-called Choose Method, he is using this notion of general element.

4. VARIABLES AND CONSTANTS

The notion of a *variable* in logic or in mathematics is a linguistic construct. A *variable ranging over a set S* is a symbol that represents a general element of S . Sometimes we want to consider several variables ranging over S (or over various sets), and for this we use separate symbols for each. You are free to use whatever symbols you like for variables, provided your usage is consistent, both within itself and with respect to other sources you may be referencing. It is common to use lower case letters near the end of the alphabet for variables—such as $r, s, t, u, v, w, x, y, z$ —but this is not mandatory. When many variables are used, their symbols often contain subscripts, as in x_1, x_2, x_3, y_1, y_2 , etc.

The notion of a *constant* in S is closely analogous to that of a variable. It is also a linguistic construct. But in this case it is a symbol used to represent a *particular* element of S , which is assumed to be held fixed for the duration of the discussion about S , or for part of the discussion. Often letters from the early part of the alphabet are used for constants, such as a, b, k, B, C , etc., but again this is not mandatory.

Sometimes notation for some constant spreads from one practitioner to another and attains wide currency, even universal acceptance. This occurs when the mathematics in question is especially important and/or the originators of the usage are very influential. In this way, symbols such as “0”, “1”, “2”, “ π ”, “ e ”, etc., have come to represent constants whose meaning is (nearly) universally understood.

5. EXPRESSIONS

Variables and constants generally appear in larger linguistic constructs, the precise nature of which depends on the mathematical system that is being considered. Usually, such a system involves *elements* in one or more sets, various *relations* among the elements of the sets, *operations* on the elements of the sets, and possibly also various kinds of *functions* relevant to the system. Some particular elements, functions, operations, or relations may be singled out and denoted by special symbols, whereas others remain general. It is often

convenient to assume that all of the sets considered in the mathematical system are subsets of some (unspecified) universal set (often denoted by some symbol such as \mathcal{U}).

Given a system with such elementary ingredients, we select symbols for representing elements, functions and operations (leaving relational symbols until later) and combine these symbols according to certain formation rules to obtain more and more complex strings of symbols that represent more complex objects of the system. Such well-formed strings are called *expressions*.

For an example of this, if one is studying integer arithmetic, one would consider the operations $+$, $-$, and \times , as well as, perhaps, the related operations of raising to the n^{th} power, for non-negative integers n . One would also want to consider the usual constants $0, \pm 1, \pm 2, \dots$ etc., as mentioned above. Thus, one would expect expressions like $x + y$, $7x + 1$, $3x^2 - 4x + 2$, $x^n - y^n$, $(x_1 + x_2 + x_3)(x_1 - x_2 + x_3)$, etc.

If one is dealing with real numbers, then one would want to include all the expressions of integer arithmetic (now extended to refer to real numbers) as well as allowing division and using function expressions. Thus, expressions such as $1/x$, $\sqrt{x^2 + y^2}$, ye^x , $\ln(1 + x)$, $\sin(xyz)$, would be adjoined to the previous list.

It is also useful to allow expressions that involve *no* variables, that is, involving only constants and operations and functions applied to these. Thus, for example, the constants $0, 1, \sqrt{2}$ are expressions, as are $1 + 1$, or e^2 , or $2\pi + 3$.

Each system has its own ingredients and rules of formation, but the general scheme of building complex expressions from simple pieces is common to all of these. All of this is defined precisely and in complete generality in a course in mathematical logic. We do not do this here, relying instead on the knowledge we have already gained from years of experience in working with mathematical expressions and with an informal understanding of the concept.

6. PREDICATES

A *predicate* P in a mathematical system is a declarative assertion involving a finite number of expressions and relations in the system. If no logical operations appear in the predicate, i.e., it contains only expressions and relations, we may say that P is a *simple predicate* or an *atomic predicate*. Just as in the case of statements in propositional calculus, more complex predicates are constructed by applying logical operations to simpler ones, starting with atomic predicates.

The variables in the expressions are drawn from a finite list, say x_1, x_2, \dots, x_n , so we may write $P = P(x_1, x_2, \dots, x_n)$ to emphasize the role of the variables. Not every expression in P need make use of every variable. We assume that x_1, x_2, \dots, x_n range over various sets: x_i ranges over the set S_i . It could be that all of the S_i are equal to one given set S , or it could be that some or all of them are distinct from the others. This depends on the mathematical system we are considering and on the predicate P . We sometimes say that P is a predicate over S_1, S_2, \dots, S_n .

For example, in the theory of real numbers, we have simple predicates such as $e^x > 0$, $x^2 + y^2 = 1$, $\sin(x + y) = \sin x \cos y + \cos x \sin y$, and so on. In number theory, we have predicates such as $m|n$, $\ell + m \equiv n \pmod{17}$, $m > n$, and so on. In the first example, x and y are real variables, $e^x, \sin x, \cos y$, etc. are well-known functions, and the relations are $>$ and $=$. In the second example, ℓ, m, n are variables and $| \equiv \pmod{17}$ and $>$ are relations.

It is assumed that a predicate P is a meaningful assertion when each variable x_i is interpreted as a general element of set S_i , $i = 1, 2, \dots$, etc. For example, $x^2 + 2x - 3 = 0$ defines a predicate $P(x)$ over the real numbers. Note that it asserts something: namely, that the square of a real number plus twice that number minus 3 equals 0. However, *it is not a statement*, because its truth or falsity depends on the particular value of x . (As already noted, Solow calls this a conditional statement).

The variables appearing in a predicate are of two kinds: those that have been quantified and those which have not. We describe quantification later. For now, we note only that quantified variables are often called *bound variables*, whereas unquantified ones are called *free variables*.

7. LOGICAL OPERATIONS AND PREDICATES

The logical operations of propositional calculus—negation, conjunction, disjunction, implication, identity, together with their iterations—can be applied to predicates just as they are to statements. Thus, for example, let $P = P(x_1, x_2, \dots, x_k)$ and $Q = Q(y_1, y_2, \dots, y_\ell)$ be predicates. We allow the possibility that some of the y_j 's equal some of the x_i 's. Then we can form new predicates

$$\neg P, P \wedge Q, P \vee Q, P \Rightarrow Q, P$$

involving the variables

$$\{x_1, \dots, x_k\} \cup \{y_1, \dots, y_\ell\}.$$

As stated in §6, these predicates are to be understood as declarative assertions about general elements of the sets over which the variables range. Note that these logical operations *do not* reduce the number of variables.

8. SPECIALIZATION

Because predicates generally have variables, they allow logical operations that are not available to us in the case of statements. (More precisely, they are trivial in the case of statements.) The simplest such operation is that of *specialization of a variable*.

To describe this precisely, we begin with the idea of *specializing a variable in an expression*. Suppose that E is an expression in which the variable x appears, x ranging over the set S . We may then write $E(x)$ to emphasize the appearance of x in E (although other unexhibited variables may also appear in E). We shall say that x is *specialized to a value a in E* if every occurrence of x in E is replaced by the same symbol a , which represents a particular element

S . (Note that it is important that this symbol *not* also be used in a different role in the expression!) A slightly different but similar locution is sometimes used for the same thing: namely, we may say that E is *specialized at* $x = a$.

If x is specialized to a in E , we may then denote the specialized expression by $E(a)$. If E is, indeed, an expression involving x and other variables y, z, \dots , then $E(a)$ is simply an expression involving y, z, \dots , but no longer involving x . If E involves only x at the outset, then $E(a)$ is an expression involving no variables, i.e., it represents a specific element of S or of some other set considered in the system.

For example, suppose S is the set \mathbb{Z} of integers and $E = E(x)$ is the expression $y - x^2 - 2x$. Then $E(1) = y - 3$. For another example, suppose S is the set of natural numbers $\mathbb{N} = \{1, 2, 3, \dots\}$, T is the set of positive reals, and $E(x) = e^{-x}$, where x ranges over \mathbb{N} . Then, $E(3) = e^{-3} = (1/e)^3 \in T$. And so on.

Often this process is described as *evaluating* $E(x)$ at $x = a$.

Of course, *we may specialize more than one variable at a time*, as long as we take care to substitute each occurrence of the variables by the specific elements to which they are being specialized. The elements may or may not be distinct from each other, as the case may dictate.

For example, when $E = y - x^2 - 2x$, as above, we may write E as $E(x, y)$ and then specialize to $E(-2, 0) = 0$ or to $E(1, 1) = -2$, etc.

If E involves exactly n variables, say x_1, x_2, \dots, x_n , then it may be convenient to exhibit all of these in our notation, and we write $E = E(x_1, x_2, \dots, x_n)$.

We now consider a predicate P in which a free variable x appears. That is, x appears as an unquantified variable in one or more expressions that appear in P . We may write $P(x)$ to focus attention on x . Let a be a particular element of the set S over which x ranges. We say that x is *specialized to* a in P if x is specialized to a in *every* expression in P that contains x . (Sometimes, instead, we may say that P is specialized at $x = a$.) We may write the result as $P(a)$.

This is now a predicate in which x no longer appears: it is a predicate in one fewer free variable, and we often say that the variable has been *eliminated*. If all the free variables in a predicate get eliminated, then, as we shall see, the result is a statement, with a definite truth-value.

This process may clearly be repeated any number of times, as long as there are variables left. For example, consider the simple expression $E(x, y) : y - x^2 - 2x$ mentioned above, and form the predicate $P(x, y) : y - x^2 - 2x = 3$. Then, $P(5, y)$ is the predicate $y - 5^2 - 2 \cdot 5 = 3$, and $P(5, 17)$ is the (false) statement $17 - 5^2 - 2 \cdot 5 = 3$.

Much the same thing will be seen in the case of quantification.

9. LOGICAL EQUIVALENCE OF PREDICATES

Just as with statements in Propositional Calculus, we are interested in when two predicates P and Q can be reasonably said to be logically equivalent. This motivates the following considerations.

Two preconditions clearly make sense. First of all, the predicates should both be part of the same mathematical context or theory (e.g., calculus, number theory, linear algebra, etc.). Secondly, both should involve the same free variables ranging over the same sets. (If the symbols used for the free variables in one predicate are different from those in the other, we assume that the variables in one of them can be re-labeled so that they are the same as corresponding variables in the other.)

With those preconditions satisfied, we can now define what it means for P and Q to be logically equivalent. For notational simplicity, we assume that the only free variables involved are x and y , so that P and Q may be written as $P(x, y)$ and $Q(x, y)$, respectively.

Then, $P(x, y)$ and $Q(x, y)$ are logically equivalent—written $P \iff Q$ —if and only if, for all possible specializations $x = a$ and $y = b$, the truth value of the statement $P(a, b)$ equals the truth value of $Q(a, b)$. (Recall, in the chapter on the propositional calculus, we have seen that $P(a, b)$ and $Q(a, b)$ have the same truth value if and only if $P(a, b) \iff Q(a, b)$ is true,

which we write more briefly as $P(a, b) \iff Q(a, b)$. So we may abbreviate the definition of logical equivalence of P and Q as follows: $P \iff Q$ if and only if $P(a, b) \iff Q(a, b)$, for all specializations $x = a$ and $y = b$.)

Here are two simple examples. In both cases the variables are assumed to range over the real numbers. (a) Let $P(x, y)$ be the predicate $\sin(x) = \sin(y)$, and let $Q(x)$ be the predicate $x = y$. Clearly, the statements $P(a, a)$ and $Q(a, a)$ are both true, for every real number a . However, $P(0, \pi)$ is true, whereas $Q(0, \pi)$ is false. Therefore, it is not the case that $P \iff Q$. (b) Let $P(x)$ be the predicate $x^2 - 1 = 0$, and let $Q(x)$ be the predicate $(x = 1) \vee (x = -1)$. It is easy to see that both $P(x)$ and $Q(x)$ are true when x equals 1 or -1 , and both are false otherwise. So $P \iff Q$.

Thus the notion of logical equivalence of predicates is based on that used for statements. And, therefore, not surprisingly, it has most of the same properties of the earlier notion. For one example, if we have a complex predicate constructed by applying logical operations to simpler predicates, and we substitute some of the simpler predicates by ones that are logically equivalent, then the resulting complex predicate is logically equivalent to the original one. We let the reader check a few simple examples to convince himself/herself of this.

10. QUANTIFICATION

Quantification is a logical operation applied to predicates P . The general scheme of things goes like this. A variable x is selected, and we quantify P with respect to x . If x does not appear in P , then the quantification operation is trivial: we get P again. If x does appear in P , then we get a new predicate in which x has been eliminated. We can now, if we choose, quantify this new predicate again with respect to some other variable. And so on.

In contrast to the operation of specialization, in which several variables may be specialized at once and the order of specialization is irrelevant, quantification is applied to one variable at a time and the order of quantification, in general, does make a difference—as we shall see in the next section.

In what follows, we must distinguish two kinds of variables in P : namely, those variables in P that we have already quantified earlier and those we have not. The former are called *bound variables*, the latter *free variables*. Quantification may be applied only to free variables, hence, no variable gets quantified more than once.

We now become more precise and describe the two kinds of quantification: *universal* and *existential*.

To fix notation, we suppose that P is a predicate containing a free variable x ranging over a set A . It is possible that P has other variables as well, some bound, some free, but to signify our present interest in x , we shall write P as $P(x)$. The so-called *scope* of x consists of all occurrences of x in P .

To quantify $P(x)$ with respect to x , we consider all possible specializations $x = a$ in P , each one giving a predicate $P(a)$ in the remaining variables. Quantification with respect to x is an assertion about these $P(a)$. There are two cases.

10.1. Universal quantification. For universal quantification, we form $(\forall x)P(x)$ (to be read “For all x , $P(x)$ ”). This asserts *all* the specialized $P(a)$.

If P contains free variables other than x , say y, z, \dots , then $(\forall x)P(x)$ is a *predicate in these remaining free variables*, and again we say that the variable x has been *eliminated*. If x is the only free variable in P , then $(\forall x)P(x)$ has no free variables, so it is a statement. Indeed, it is precisely the statement that asserts *all of the statements $P(a)$ simultaneously*.

Here are two examples:

- Let P be the predicate $x^2 + 1 \neq 0$, and suppose that x ranges over the set of real numbers. Then, $(\forall x)P(x)$ asserts: for every real number a , $a^2 + 1 \neq 0$. This is certainly a true statement.
- Let P be the predicate $3x + 4y = 5$, so that P has two variables x and y . We suppose that they both range over the set of rational numbers. Then, $(\forall x)P(x)$ is the assertion: for each rational number a , $3a + 4y = 5$. This is a predicate involving one variable

y . If y is subsequently specialized to some rational value, say b , then the predicate becomes a *statement* asserting that, for each rational number a , $3a + 4b = 5$. Clearly, such an assertion is false no matter what specific rational number b represents.

*Therefore, to repeat: With universal quantification we are asserting **all** of the specialized assertions $P(a)$ at once. When x is the only free variable in P , we see that $(\forall x)P(x)$ is a true statement precisely when all of the $P(a)$ are true simultaneously.*

10.2. Existential quantification. For existential quantification, we form $(\exists x)P(x)$ (to be read, “There exists an x such that $P(x)$ ”). This is the predicate that asserts at least one of the predicates $P(a)$. (Note that it does not specify which ones.)

Again, if P contains other free variables y, z, \dots , then $(\exists x)P(x)$ is a predicate in these remaining free variables. If not, then $(\exists x)P(x)$ is the statement that “At least one of the statements $P(a)$ is true.”

Here are two examples of existential quantification:

- Let P be the predicate $x^2 + x + 1 = 0$, where x ranges over the real numbers. Then $(\exists x)P(x)$ asserts that the polynomial has a real root, which the student may easily check to be a false statement.
- Let P be the predicate $x^2 + y^2 + 2xy - y + 3 = 0$, where both x and y range over the reals. Then $(\exists x)P(x)$ is a predicate in the variable y . We may also write this as $(\exists x)P(x, y)$ to signal the role that y plays. In any case, let us write this new predicate as $Q(y)$. For any specialization $y = b$, $Q(y)$ clearly becomes $Q(b)$ —i.e., $(\exists x)P(x, b)$ —which is the statement that the polynomial $x^2 + b^2 + 2xb - b + 3 = 0$ has a real root. Using the quadratic formula, it is not hard to show that this statement is true when $b \geq 3$, and it is false when $b < 3$.

*Therefore, to repeat: With existential quantification we are asserting **at least one** of the specialized assertions $P(a)$ (but not specifying which one). When x is the only free variable*

in P , we see that $(\exists x)P(x)$ is a true statement precisely when at least one specialization $P(a)$ is a true statement.

Exercise 1. Verify the last assertion in the second example above.

11. CHANGING THE ORDER OF QUANTIFICATION

When we apply logical operations successively to predicates (or read a predicate that involves a succession of logical operations), we must be careful about the order in which the operations are applied. This is, for example, completely analogous to composing linear operators defined on a vector space, or to multiplying matrices. The order in which this is done will affect the answer.

In this section, we look at some examples of this in case the operations we exchange are both quantification operations. The following two sections show what happens when we try to exchange quantification with some of the elementary logical operations.

For our first example, let us deal with real variables again, and let R be the predicate $x + y < 1$. Now consider the statement $(\forall x)(\exists y)R(x, y)$. To determine whether it is true or false, we begin at the left, just as with function composition. We must decide whether, for every real number r , the statement $(\exists y)R(r, y)$ is true. Now, in turn, this statement is true if we can find at least one real number s such that $R(r, s)$ is true: that is $r + s < 1$. Clearly, this last is possible: just choose s to be any real number $< 1 - r$. So, we have shown that $(\forall x)(\exists y)R(x, y)$ is true.

But notice what happens when the quantifiers are interchanged. We obtain the statement $(\exists y)(\forall x)R(x, y)$. Just by reading the statement —“There is a y such that for all x , $R(x, y)$ ”—one can see that this is not the same as the previous statement. But, to clinch the matter, let us check its truth value. Again start on the left. To determine whether or not the statement is true, we must find at least one real number u such that $(\forall x)R(x, u)$ is true. And to demonstrate this last, we must show that for any real number v , $v + u < 1$. But no matter

what our choice of u , the real number $v = 2 - u$ violates the condition $v + u < 1$. Therefore, the statement $(\exists y)(\forall x)R(x, y)$ is false.

There are, however, certain cases of successive quantification in which the order does not matter. Here is a rule that covers these cases: *Whenever two successive quantification are of the same type—i.e., both universal or both existential—then the order of quantification does not matter.*

For example, consider the predicate $P(x, y) : x^2 + 2y - 6 < 0$, defined for x and y ranging over the real numbers. Then $(\exists x)(\exists y)P(x, y)$ is logically equivalent to $(\exists y)(\exists x)P(x, y)$. For each of these is true exactly when $P(a, b)$ is true for some real numbers a and b (in fact, precisely when the real numbers a and b satisfy $a^2 < 6 - 2b$) and false otherwise. For another example, let $Q(x, y)$ be the predicate $e^{x+y} = e^x \cdot e^y$, where x and y range over the reals. Then $(\forall x)(\forall y)Q(x, y)$ and $(\forall y)(\forall x)Q(x, y)$ both assert (truthfully in this case) that the statement $Q(a, b)$ is true for all real numbers a and b .

12. THE INTERACTION BETWEEN QUANTIFICATION AND NEGATION

Let P be a predicate containing the free variable x ranging over the set S (and perhaps other variables). Then, the basic facts relating $\forall x$ to negation can be stated as two propositions:

1. Proposition. $\neg(\forall x)P(x) \iff (\exists x)(\neg P(x))$.

2. Proposition. $\neg(\exists x)P(x) \iff (\forall x)(\neg P(x))$.

Exercise 2. Prove each of the two propositions under the assumption that x is the only free variable in P . (Hint: Note that in this special case, the two predicates exhibited in the first proposition are actually statements. So, it suffices to show that these two statements have the same truth value. This can be derived directly from the definition of quantification. Similarly for the second proposition.)

Notice that in both propositions, on the left-hand side, we are negating first and then quantifying, whereas on the right-hand side we are quantifying first and then negating.

Therefore, *to make a valid change of order in this case, we have to change the quantifier*: in the first proposition, we change the universal quantifier to an existential one; in the second proposition, we do the reverse.

Here are some special cases of the above propositions. In the first four, De Morgan's laws are also used. In the last two, one also makes use of the logical equivalence $\neg(A \Rightarrow B) \iff A \wedge \neg B$.

A word of advice (which applies to many rules and formulas in mathematics): It is best not to try to memorize the following list. Rather, remember the principles by which the list is constructed. In this case there are two principles: (i) Exchanging the order of quantification and negation changes a universal quantifier to an existential and an existential quantifier to a universal. (ii) Exchanging negation with an elementary logical operation follows the rules in the propositional calculus. Specifically, De Morgan's Laws show that a disjunction is changed to a conjunction and vice versa.

- $\neg(\forall x)(A(x) \wedge B(x)) \iff (\exists x)\neg(A(x) \wedge B(x)) \iff (\exists x)(\neg A(x) \vee \neg B(x))$.
- $\neg(\forall x)(A(x) \vee B(x)) \iff (\exists x)\neg(A(x) \vee B(x)) \iff (\exists x)(\neg A(x) \wedge \neg B(x))$.
- $\neg(\exists x)(A(x) \wedge B(x)) \iff (\forall x)\neg(A(x) \wedge B(x)) \iff (\forall x)(\neg A(x) \vee \neg B(x))$.
- $\neg(\exists x)(A(x) \vee B(x)) \iff (\forall x)\neg(A(x) \vee B(x)) \iff (\forall x)(\neg A(x) \wedge \neg B(x))$.
- $\neg(\forall x)(A(x) \Rightarrow B(x)) \iff (\exists x)(A(x) \wedge \neg B(x))$.
- $\neg(\exists x)(A(x) \Rightarrow B(x)) \iff (\forall x)(A(x) \wedge \neg B(x))$.

To be more concrete, let us take the third of these, letting $A(x)$ be the predicate $x^2 - 3x + 2 = 0$ and $B(x)$ the predicate $x^2 - 6x + 9 = 0$. Then $\neg\exists(x)(A(x) \wedge B(x))$ asserts that it is not the case that the two polynomial equations have a root in common, whereas $(\forall x)(\neg A(x) \vee \neg B(x))$ asserts that, for every real number x , either $x^2 - 3x + 2 \neq 0$ or $x^2 - 6x + 9 \neq 0$. Clearly these two statements assert the same thing, so they have the same truth-value.

13. THE RELATIONSHIP BETWEEN QUANTIFICATION AND CONJUNCTION AND
DISJUNCTION

We begin with two cautionary examples.

First, suppose that $A = A(x)$ is the predicate $x^2 + 3x + 2 = 0$ and $B = B(x)$ is the predicate $4x - 7 = 0$, where the variable x ranges over, say, the complex numbers \mathbb{C} . Now consider the following statements:

$$(1) \qquad (\exists x)(A(x) \wedge B(x))$$

$$(2) \qquad (\exists x)A(x) \wedge (\exists x)B(x).$$

Statement (1) asserts the existence of a real number which is a *simultaneous* root of the two equations, whereas statement (2) asserts only that each equation separately has a root. Clearly, these are two very different assertions, so it is not surprising that they have different truth values, statement (1) being false and statement (2) true.

Next, we'll use the predicates $C(x) : x < 3$ and $D(x) : x > 0$, where x is a variable ranging over the reals, and we consider

$$(3) \qquad (\forall x)(C(x) \vee D(x))$$

$$(4) \qquad (\forall x)C(x) \vee (\forall x)D(x)$$

In statement (3), we are first performing a disjunction and then quantifying; in (4), we are reversing the order. Statement (3) asserts that every real number is either less than 3 or greater than 0, which is certainly true. Statement (4) asserts *both* that every real number is < 3 *and* that every real number is > 0 . Neither of these is true, so certainly statement (4) is false.

If we look closely at what “went wrong” in these examples, we can see that the problem involves the *scope* of the quantification operations. In statement (1), there is one instance of quantification over the variable x , the scope of which involves both predicates A and B . In statement (2), there are two quantification operations, each with more limited scope (the one involving only A , the other only B). A similar observation applies to statements (3) and (4).

These examples show that we have to be careful about the scope of the quantifiers before trying to exchange them with conjunction or disjunction.

The observation about scope, however, shows us how we may do this in cases like those of statements (2) and (4). Let $\mathcal{Q}x$ denote either $\forall x$ or $\exists x$, and let $E(x)$ and $F(x)$ be two predicates involving the free variable x (and perhaps some other variables). We’ll make use of the fact that we may change the name of the variable x without affecting truth-values. We can choose any symbol instead of x provided it does not already appear in the predicate. So, suppose y does not appear in F , and then replace x by y . Then, $(\mathcal{Q}x)F(x)$ asserts the same thing as $(\mathcal{Q}y)F(y)$. Now make sure that y is chosen not only to be different from any variable appearing in F originally, but also different from any variable appearing in E . Then, the two predicates

$$(5) \quad (\mathcal{Q}x)E(x) \wedge (\mathcal{Q}x)F(x), \text{ and}$$

$$(6) \quad (\mathcal{Q}x)E(x) \vee (\mathcal{Q}x)F(x)$$

can be replaced by

$$(7) \quad (\mathcal{Q}x)E(x) \wedge (\mathcal{Q}y)F(y), \text{ and}$$

$$(8) \quad (\mathcal{Q}x)E(x) \vee (\mathcal{Q}y)F(y),$$

respectively, without changing their meaning.

Having done this, we can now see that, for each predicate, since the two quantifications involve different variables, we may move the second quantifier to the left, without affecting the meaning or truth value):

$$(9) \quad (\mathcal{Q}x)(\mathcal{Q}y)(E(x) \wedge F(y)), \text{ and}$$

$$(10) \quad (\mathcal{Q}x)(\mathcal{Q}y)(E(x) \vee F(y)).$$

Therefore, a quantifier may be “moved to the left”, provided the quantified variable is given a new name different from those of other variables already appearing.

Note that if we apply this to statements (2) and (4), we do get all the quantifiers on the left, but there are still two quantifications required rather than just one as in statements (1) and (3). Statements (1) and (3) (which already have their single quantifier on the left) are not affected by this procedure.

The above procedure still works even when the second quantifier is not the same as the first. For example, we can replace $(\exists x)E(x) \wedge (\forall y)F(y)$ by $(\exists x)(\forall y)(E(x) \wedge F(y))$, etc.

Finally, the logical equivalence (f) in Exercise 11 of the propositional calculus notes

$$(P \Rightarrow Q) \quad \Longleftrightarrow \quad (\neg P \vee Q),$$

together with the fact that substituting a predicate by a logically equivalent one does not affect truth values, shows that the relationship between quantification and implication may be understood in terms of relationship between quantification, negation, and disjunction. We don't go any further into this, but we present the following exercise as an illustration.

Exercise 3. Let $P(x, y)$ be the predicate $x < y^{-1}$ and $Q(x, z)$ the predicate $-x = z^2$. Here we assume that x and z range over the real numbers and y ranges over the *positive* real

numbers. Show that

$$\left((\forall y)P(x, y) \Rightarrow (\exists z)Q(x, z) \right) \iff (\exists y)(\exists z)\left(P(x, y) \Rightarrow Q(x, z) \right).$$

Exercise 4. Formulate the following statements using predicates and quantifiers:

- (a) If a, b , and c denote the lengths of the sides of a planar triangle, then $a + b > c$.
- (b) If a, b , and c denote the lengths of the sides of a planar right triangle, with c the largest, then $a^2 + b^2 = c^2$.
- (c) If a, b, c and n are any positive integers and $n > 2$, then $a^n + b^n \neq c^n$.

Now reformulate each of the above, if necessary so that only negation, disjunction, and quantification are used.

Statement (a) is a special case of the well-known Triangle Inequality that students often see in a linear algebra course. Statement (b) is, of course, the Pythagorean Theorem, which is one of the cornerstones of geometry. And statement (c) is known as Fermat's Last Theorem. It was proved in 1994 by Andrew Wiles, more than 350 years after it was conjectured by Pierre de Fermat.

- Exercise 5.** (a) Let f be a real-valued function defined on an open interval I of real numbers, and let a be an element of I . The following statement expresses the fact that f is *continuous at* a . Formulate that statement in terms of quantifiers and predicates: For every $\epsilon > 0$, there is a $\delta > 0$ such that, for every real x satisfying $0 < |x - a| < \delta$, we have $|f(x) - f(a)| < \epsilon$. Now formulate this so that only quantification, negation and disjunction are used and so that all quantification occurs on the left.
- (b) Let f be a real-valued function defined on an open interval I of real numbers, and let a be an element of I , as in the foregoing. Express the assertion that f is *not* continuous at a by applying negation to the last statement you obtained in the preceding exercise and exchanging negation with the other operations according to the rules we've derived. In the end, you should have a statement for which negation has been

completely “distributed” (i.e., cannot be exchanged any further with other operations without reversing steps).

- (c) Let f be a continuous real-valued function defined on a closed interval J . Use quantifiers (on the left) and predicates to express the fact that f achieves a maximum value on J .
- (d) Suppose that g is a differentiable (hence continuous) real-valued function defined on an open interval I . Use quantifiers and predicates to express the fact that if g achieves a maximum value at some element $x \in I$, then the derivative $f'(x) = 0$.

This concludes our brief foray into the foundations of the predicate calculus. For more advanced work in this area, consider taking Math 481 or Math 483.