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Matroids and quotients of spheres

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Abstract. For any linear quotient of a sphere, $X = S^{n-1}/\Gamma$, where Γ is an elementary abelian p–group, there is a corresponding $\mathbb{F}_p$ representable matroid M_X which only depends on the isometry class of X. When p is 2 or 3 this correspondence induces a bijection between isometry classes of linear quotients of spheres by elementary abelian p–groups, and matroids representable over $\mathbb{F}_p$. Not only do the matroids give a great deal of information about the geometry and topology of the quotient spaces, but the topology of the quotient spaces point to new insights into some familiar matroid invariants. These include a generalization of the Crapo–Rota critical problem inequality $\chi(M; p^k) \geq 0$, and an unexpected relationship between $\mu(M)$ and whether or not the matroid is affine.

1 Introduction

Linear actions on spheres by finite groups can be analyzed via representation theory. The corresponding quotient spaces have been largely ignored. While space forms have been extensively studied by geometers, both before and after they were classified by Wolf [23], they are only a tiny fraction of all finite linear quotients of spheres. Many of the basic geometric and topological properties of these spaces are unknown. For instance, let G be a finite group. Suppose G acts by diffeomorphisms on an n–dimensional manifold $\mathcal{M}^n$. When is $\mathcal{M}^n/G$ a topological manifold? At a minimum, for each point $x \in \mathcal{M}^n$ the spherical quotient corresponding to the isotropy representation of G_x on the unit tangent sphere at x must have the homology of S^{n-1}. When does this happen? While the representation theory of G_x may give an excellent description of the isotropy action, the corresponding quotient space is still a mystery.

In this paper we will examine the close connection between binary matroids and linear quotients of spheres by elementary 2–groups. We call the latter binary spherical quotients (BSQ's). Not only will the matroids provide a great deal of information about the geometry and topology of BSQ's, but the topology of the spaces will point to new insights into some familiar matroid invariants.

Matroid theory was introduced by Whitney as a combinatorial abstraction of linear independence [22]. Since their introduction, matroids have appeared in a wide variety of settings including graph theory, hyperplane arrangements and linear coding theory. See [21] for several more applications. Binary matroids are those matroids which come from finite subsets of vector spaces over $\mathbb{F}_2$.

Section 2 contains the definitions and facts we need from geometry and topology. While we will make no use of them, our point of view is heavily influenced by Alexandrov spaces with curvature bounded below by one. (See [6] for a definition.) It was in this context that linear quotients of the form $S^{n-1}/(\mathbb{Z}_2)^r$ were, to a large extent, first analyzed.

After establishing how to classify linear quotients of spheres for arbitrary finite groups in Sect. 3, we introduce the necessary matroid theory in Sect. 4. This includes the Tutte polynomial, characteristic polynomial and the Möbius invariant of a matroid. Originally introduced by Tutte for graphs ([18],[19]), and extended to matroids by Crapo in [9], the Tutte polynomial is a two–variable polynomial matroid invariant which we denote by $T(M;x,y)$, where M is the matroid. Our primary interest will be in the specializations $T(M;0,y)$ and $T(M;x,0)$. These specializations are closely related to the characteristic polynomial, $\chi(M;t)$, of a matroid introduced by Rota [14]. All of these invariants have a substantial literature and several applications. We refer the reader to the surveys [3], [5], [25].

Let X be a BSQ. The corresponding matroid M_X is defined using either representation theory or the geometry of X. The matroid properties of M_X have natural geometric translations in X. We only give those translations necessary for the rudimentary structure theory of BSQ's in Sect. 5 and the later homological computations in Sects. 7–9. See [16] for a more extensive exploration of what is possible in this direction.

While the binary matroid theory of M_X is useful in understanding X, the homology of X sheds light on $T(M_X;0,t)$, $|\chi|(M^\star_X;t)$, and the Möbius invariant of M. Here, unlike previous homological interpretations of matroid invariants, *torsion* plays a key role. The $\mathbb{Z}_2$–Betti numbers of $H_\star(X,\mathbb{Z})$ lead us to Theorem 8, a new inequality for matroids representable over any finite field. This inequality is a generalization of the Crapo–Rota critical problem inequality $\chi(M;|F|^k) \geq 0$. In addition, the Euler characteristic of X points to a previously unnoticed relationship between the Möbious invariant of M and whether or not the matroid is affine (Theorem 9).

The main tool in the proof of the formulas for $H_\star(X)$ is an algebraic subchain complex, $\tilde{\Delta}_\star(X)$, of $\Delta_\star(X)$ a CW–chain complex for X. The Poincaré polynomial of $\tilde{\Delta}_\star(X)$ is $t^{r-1}|\chi|(M_x^\star;t)$, and the $\mathbb{Z}_2$ Poincaré polynomial of X is $t^{r-1}T(M_X^\star;t,0)$. The proof of Proposition 9 shows that if ϕ is the inclusion map of $\tilde{\Delta}_\star(X)$ into $\Delta_\star(X)$, then with $\mathbb{Z}_2$ coefficients $\phi_\star$ is surjective. This is not surprising since the coefficients of $|\chi|(M;t)$ are greater than or equal to the corresponding coefficients of $T(M;t,0)$ for any matroid M. However, previous combinatorial interpretations of this fact (for instance, the broken circuit complex [5]) are dependent on ordering the elements of M. This "natural surjection" phenomenon will recur when in the future we examine the relationship between $T(M_X;1,t)$ and the cohomology of the manifold consisting of the regular points of X.

We use $G_\mathbb{R}$ for the set of equivalence classes of irreducible real representations of G. Two representations $\rho_i : G \to GL(n,\mathbb{R}), i \in \{1,2\}$ are equivalent if there exists $T \in GL(n,\mathbb{R})$ such that $T \circ \rho_1 \circ T^{-1} = \rho_2$. A representation $\rho : G \to GL(n,\mathbb{R})$ is *faithful* if ρ is 1–1. If A is a set and $e \notin A$, then $A \cup \{e\}$ is abbreviated $A \cup e$. Similarly, $A - e$ stands for $A - \{e\}$. In order to allow for irreducible representations with multiplicities and vector representations of matroids, *subsets of $G_\mathbb{R}$ and subsets of vector spaces may contain repeated elements.*

2 Geometric notions

Let $X = S^{n-1}/G$, where the group G is finite and acts on the left by isometries. The metric on X is determined by setting the distance between two orbits Gx and Gy to be the minimum of $d(gx, g'y)$ over all g, g' in G. In general X will have singularities but, except for $S^0/\{id\}$, it will still be a *length space*. That is, the distance between two points in X is the infimum of the lengths of all paths between the two points.

A *geodesic* in X is a path which is locally length minimizing. We always assume that geodesics are parameterized by arc length. If c is a geodesic from x to y such that the length of c equals $d(x,y)$, then we say that c is a *segment*. A subset Y of X is *totally convex* if every segment in X between any two points in Y is contained in Y. The *convex hull* of Y is the smallest totally convex subset which contains Y. The *diameter* of X is the maximum of $d(x,y)$, x and y points in X. The (spherical) suspension of X is denoted ΣX. As a topological space ΣX is $X \times [0,\pi]/(x,0) \sim (y,0) \cup (x,\pi) \sim (y,\pi)$. The metric on ΣX is uniquely determined by insisting that for any segment c of length $l(c)$ in X, the image of $c \times [0,\pi]$ in ΣX is isometric to the region swept out between the north and south poles in the unit sphere S^2 by an angle of measure $l(c)$. If x is a fixed point of G, then $-x$ is also a fixed point, G acts on S_x^{n-1}, where $S_x^{n-1} = \{y \in S^{n-1} : d(x,y) = \pi/2\}$, and S^{n-1}/G is isometric to $\Sigma(S_x^{n-1}/G)$. The cone on X is denoted by CX and is the image of $X \times [0,\pi/2]$ in ΣX.

Let Y be another length space of diameter less than or equal to π. The (spherical) join of X and Y is $X * Y$. As a topological space $X * Y$ is $X \times [0,\pi/2] \times Y/\sim$, where $(x,0,y) \sim (x,0,y')$ and $(x',\pi/2,y) \sim (x,\pi/2,y)$ for all x, x' in X, and y, y' in Y. In order to define the metric on $X * Y$, let c be a segment in X, c' a segment in Y and let $l(c), l(c')$ be their respective lengths. Let γ and γ' be segments in S^3 such that $l(\gamma) = l(c), l(\gamma') = l(c')$, and the distance from any point in γ to any point in γ' is $\pi/2$. Let $\gamma * \gamma'$ be the union of all segments from points in γ to points in γ'. Then the metric on $X * Y$ is determined by insisting that the obvious map of the image of $c \times [0,\pi/2] \times c'$ in $X * Y$ to $\gamma * \gamma'$ is an isometry.

We use S_x for the space of directions at x in X. It is the set of germs of segments beginning at x with angle measure as a metric (see [6] for details). It is isometric to S_x^{n-1}/G_x, where $G_x = \{g \in G : g(x) = x\}$ is the isotropy group of G at x. We say x is a *regular point* of X if $G_x = \{1_G\}$.

3 Linear quotients of spheres

If Γ and Γ' are conjugate subgroups of $O(n)$, then their quotients are isometric. What about the converse? A trivial counterexample with infinite subgroups is $S^1/O(2) = S^1/SO(2) = \{*\}$. For finite groups we have the following lemma which is closely related to universal orbifold covering spaces ([17, chap. 13]).

Lemma 1. *Let Γ and Γ' be finite subgroups of $O(n)$. If S^{n-1}/Γ is isometric to S^{n-1}/Γ', then Γ is conjugate to Γ' in $O(n)$.*

Proof. Let x be a regular point of $X = S^{n-1}/\Gamma$. Let U_x be a neighborhood of x isometric to a small metric ball in S^{n-1} such that the inverse image of U_x consists of $|\Gamma|$ distinct components isometric to U_x. Let N be the north pole of S^{n-1} and let U_N be a neighborhood N isometric to U_x such that there is an isometry $\psi : U_N \to U_x$ with $\psi(N) = x$. Now, Γ is conjugate to a subgroup Λ such that the quotient map $P_\Lambda : S^{n-1} \to X$ restricted to U_N is ψ. The lemma is equivalent to the statement that Λ and P_Λ are uniquely determined by the requirement that P_Λ restricted to U_N is ψ. This is done by induction, with the initial case $n = 1$ being trivial.

Let c be an arbitrary geodesic emanating from the north pole. As long as $P_\Lambda(c)$ remains in the regular part of X, $P_\Lambda(c(t))$ is determined by the fact that $P_\Lambda(c)$ is still a geodesic in X. Let $y = P_\Lambda(c(t))$ be a singular point in X. How do we know how to continue $P_\Lambda(c)$ beyond y?

Let S_y be the space of directions of y in X. The induction hypothesis implies that however we identify S_y with S^{n-2}/Λ_y, the exit direction of $P_\Lambda(c)$ will be the same. So the continuation of $P_\Lambda(c)$, and hence also P_Λ, is uniquely determined.

Since x is a regular point of X, each preimage of x corresponds to exactly one element of Λ (the north pole representing the identity). Any $\lambda \in \Lambda$ is completely determined by the requirement that $\lambda : U_N \to \lambda(U_N)$ covers $P_\Lambda : U_N \to U_x$ and $P_\Lambda : \lambda(U_N) \to U_x$. Hence Λ is also unique.

So, when G is finite, classifying isometry classes of quotients of S^{n-1} by faithful linear G–actions is equivalent to classifying conjugacy classes of subgroups of $O(n)$ isomorphic to G. Since images of equivalent representations are conjugate, one approach to the latter problem is through representation theory.

Precomposition gives a natural right action of $Aut(G)$ on $G_{\mathbb{R}}$. Let $Inn(G)$ be the inner automorphisms of G, and let $Out(G) = Aut(G)/Inn(G)$ be the outer automorphism group of G. Since $Inn(G)$ acts trivially on $G_{\mathbb{R}}$, the right action of $Aut(G)$ on $G_{\mathbb{R}}$ factors through $Out(G)$. From elementary representation theory there is a 1–1 correspondence between finite dimensional representations of G and finite subsets of $G_{\mathbb{R}}$ [15]. (Recall our convention concerning subsets of $G_{\mathbb{R}}$.) Let $FG_{\mathbb{R}}$ be the subsets of $G_{\mathbb{R}}$ which correspond to faithful representations of G. The $Out(G)$ action on $G_{\mathbb{R}}$ extends to $FG_{\mathbb{R}}$ in a canonical way.

Proposition 1. *Let G be a finite group. There is a 1–1 correspondence between isometry classes of quotients of spheres by faithful linear G–actions and the Out(G) orbits of* $FG_{\mathbb{R}}$.

Proof. Given two equivalence classes in $FG_{\mathbb{R}}$ whose images in $O(n)$ are all in the same conjugacy class we can choose two representatives, say ρ_1 and ρ_2, with identical images. Then $\rho_2^{-1} \circ \rho_1$ is an automorphism of G. Hence the original equivalence classes lie in the same $Out(G)$ orbit of $FG_{\mathbb{R}}$. On the other hand, the images of two equivalence classes in the same $Out(G)$ orbit of $FG_{\mathbb{R}}$ are all conjugate.

The proposition is false if we allow non–faithful actions. For example, let $G = \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p$. Let ρ_1 be a representation of G with kernel $\mathbb{Z}_p \oplus \mathbb{Z}_p$, and let ρ_2 be a representation of G with kernel $\mathbb{Z}/p^2\mathbb{Z}$. Even if the images are identical, there is no automorphism Φ of G such that $\rho_1 = \rho_2 \circ \Phi$.

In theory, Proposition 1 can be applied to any finite group. Of course, for an arbitrary group the $Out(G)$ orbits of $FG_{\mathbb{R}}$ may be very difficult to analyze.

Example 1. Let $G = \Sigma_m, m \neq 6,$ the symmetric group on m letters. Since Out(Σ_m) is trivial when $m \neq 6,$ the set of all spherical quotients are classified by $FG_{\mathbb{R}}$. The correspondence between $G_{\mathbb{R}}$ and partitions of m allows an easy enumeration of $FG_{\mathbb{R}}$.

Example 2. Let $G = (\mathbb{Z}_2)^r$. Now $G_{\mathbb{R}}$ consists entirely of one–dimensional representations which form a group isomorphic to $(\mathbb{Z}_2)^r$. The group operation is defined by $(\rho_1 + \rho_2)(g) = \rho_1(g) \cdot \rho_2(g)$. Let α be the group isomorphism between $\mathbb{Z}_2$ and $O(1)$. A useful identification of $G_{\mathbb{R}}$ with $(\mathbb{Z}_2)^r$ is obtained by assigning to $g \in G$ the representation $\rho_g(h) = \alpha(g \cdot h)$, where $g \cdot h$ is the usual $\mathbb{Z}_2$–inner product on $(\mathbb{Z}_2)^r$. Under this identification the outer automorphism group of G is $Aut(G) = GL(r, 2)$ and the right action is the usual right action of $GL(r, 2)$ on $(\mathbb{Z}_2)^r$.

With a minor modification to take into account the fact that the trivial representation of $(\mathbb{Z}_p)^r$ is one–dimensional while all others are two–dimensional, Example 2 can be extended to $(\mathbb{Z}_p)^r$ for odd primes p. In both cases $G_{\mathbb{R}}$ has a natural vector space structure and Out(G) preserves independence relations. To summarize: if G is an elementary abelian p–group then we can view an element of $FG_{\mathbb{R}}$ as a set of vectors in a vector space over $\mathbb{Z}_p$. Two elements of $FG_{\mathbb{R}}$ lie in the same $Out(G)$ orbit if and only if, when viewed as sets of vectors, they differ by an element of $GL(r, p)$. In the language of matroid theory, this sets up a map between the $Out(G)$ orbits of $FG_{\mathbb{R}}$ and rank r matroids representable over $\mathbb{Z}_p$.

4 Matroids

In this section we give the basic definitions and results from matroid theory that we will require. Matroid definitions and notation are as in [13]. The Tutte polynomial, characteristic polynomial, and Möbius invariant can be found in [3].

A *matroid*, M, is a pair $(E, \mathcal{I})$, E a non–empty finite set and $\mathcal{I}$ a distinguished set of subsets of E. The members of $\mathcal{I}$ are called the *independent* subsets of M and are required to satisfy:

1. The empty set is in $\mathcal{I}$.
2. If B is an independent set and $A \subseteq B$, then A is an independent set.
3. If A and B are independent sets such that $|A| < |B|$, then there exists an element $x \in B - A$ such that $A \cup x$ is independent.

Matroid theory was introduced by Whitney [22]. The prototypical example of a matroid is a subset of a vector space over a field F with the canonical independent sets. A matroid is *representable over F* if it is isomorphic to such an example. If the vectors are the column vectors of a matrix, we say that the matrix represents M. A matroid representable over $\mathbb{F}_2$ is called *binary.* Another source of matroids is graph theory. The cycle matroid of a graph is the matroid whose finite set is the edge set of the graph and whose independent sets are the acyclic subsets of edges. Most matroid terminology can be traced back to these two types of examples.

An element e of a matroid is a *loop* if it is not contained in any independent set. The *circuits* of a matroid are its minimal dependent sets. Every loop of M is a circuit. A maximal independent set is called a *basis,* and any element which is contained in every basis is a *coloop* of the matroid. Every basis of M has the same cardinality. The *rank* of M, or $r(M)$, is that common cardinality. The *deletion* of M at e is denoted $M - e$. It is the matroid whose finite set is $E - e$ and whose independent sets are simply those members of $\mathcal{I}$ which do not contain e. The *contraction* of M at e is denoted M/e. It is a matroid whose finite set is also $E - e$. If e is a loop or a coloop of M then $M/e = M - e$. Otherwise, a subset I of $E - e$ is independent in M/e if and only if $I \cup e$ is independent in M. Deletion and contraction for a subset A of E is defined by repeatedly deleting or contracting each element of A. The restriction of M to A is just $M - (E - A)$. We use $M|A$ for the restriction of M to A, but we will often just write A if M is unambiguous. The *rank* of A is $r(M|A)$. We either use $r(A)$ or $r_M(A)$ for the rank of A, depending on whether or not it is clear which matroid A is contained in. Note that $r(\emptyset) = 0$. While $A \subseteq M$ is technically incorrect, we will frequently use it instead of $A \subseteq E$, especially when the matroid structure of A is important. The same applies to $e \in M$ versus $e \in E$.

The *dual* of M is $M^\star$. It is the matroid whose finite set is the same as M and whose bases are the complements of the bases of M. If M is representable over F then so is $M^\star$. For example, $U_{i,j}$ is the matroid defined by $E = \{1, 2, \ldots, j\}$ and $\mathcal{I} = \{A \subseteq E : |A| \leq i\}$. So, $U^\star_{i,j} = U_{j-i,j}$.

Let $M = (E, \mathcal{I})$ and $M' = (E', \mathcal{I}')$ be two matroids with $E \cap E' = \emptyset$. Then $M \oplus M'$ is the direct sum of M and M'. It is the matroid whose finite set is $E \cup E'$ and whose independent sets are those subsets of the form $I \cup I', I \in \mathcal{I}, I' \in \mathcal{I}'$.

The *Tutte polynomial* of M is a two–variable polynomial matroid invariant. It is given by the formula,

$$T(M; x, y) = \sum_{A \subseteq M} (x-1)^{r(M)-r(A)}(y-1)^{|A|-r(A)}$$

The coefficient of $x^i y^j$ in $T(M; x, y)$ is non–negative and written $b_{i,j}$. The Tutte polynomial is the unique matroid invariant which satisfies deletion–contraction recursion (also called Tutte recursion):

1. If M is the matroid consisting of a single loop, then $T(M;x,y)=y$.
2. If M is the matroid consisting of a single coloop, then $T(M;x,y)=x$.
3. If e is a loop of M, then $T(M;x,y)=y\,T(M/e;x,y)$.
4. If e is a coloop of M, then $T(M;x,y)=x\,T(M-e;x,y)$.
5. If e is neither a loop nor a coloop of M, then $T(M;x,y)=T(M-e;x,y)+T(M/e;x,y)$.

The Tutte polynomial also satisfies $T(M^\star;x,y)=T(M;y,x)$. For direct sum we have the formula $T(M\oplus M';x,y)=T(M;x,y)\cdot T(M';x,y)$.

A subset of M is *closed* if adding any element to the subset increases its rank. The closed subsets of M form a partially ordered set under inclusion which we denote by L_M. Let L be any finite partially ordered set. The *Möbius function* on L is the function $\mu: L\times L\to\mathbb{Z}$ which satisfies:

1. If x and y are incomparable elements of L, then $\mu(x,y)=0$.
2. For any x in L, $\mu(x,x)=1$.
3. For any x,y in $L, x<y$, $\displaystyle\sum_{x\le z\le y}\mu(x,z)=0$.

The *characteristic polynomial* of M is defined using L_M. If the empty set is in L_M, then

$$\chi(M;t)=\sum_{A\in L_M}\mu(\emptyset,A)t^{r(M)-r(A)}.$$

If the empty set is not in L_M, then $\chi(M;t)=0$. The Möbius invariant of M is $\mu(M)=\mu(\emptyset,E)$. If the empty set is not in L_M, then $\mu(M)=0$. The Möbius invariant of M is also the constant term of the characteristic polynomial of M. In addition, $\chi(M;t)=(-1)^{r(M)}T(M;1-t,0)$.

5 Binary spherical quotients

Let $G=(\mathbb{Z}_2)^r$. We call any linear quotient of the form $S^{n-1}/\Gamma, \Gamma\subseteq O(n)$, where Γ is isomorphic to G and n and r are arbitrary, a *binary spherical quotient*, or BSQ. Binary spherical quotients where the group acts without fixed points are studied in [11] where they are called spaces of maximal extent. Every BSQ is either a space of maximal extent or a (possibly multiple) suspension of such a space. Spaces of maximal extent (and hence BSQ's) can be completely characterized by their metric properties ([11, Theorem 2.13]).

Let $X=S^{n-1}/\Gamma$ be a BSQ. Let $\rho: G\to\Gamma$ be a faithful representation. The corresponding subset of $FG_{\mathbb{R}}$ represents a binary matroid which we denote by M_X. The matroid isomorphism class of M_X is unchanged by the action of $GL(r,2)$ and hence only depends on the isometry class of X.

Example 3. Let $X = \mathbb{R}P^{n-1}$. Here, $G = \mathbb{Z}_2$ and the corresponding subset of $G_\mathbb{R}$ consists of n copies of the non–trivial representation of $\mathbb{Z}_2$. So, M_X equals $U_{1,n}$, the matroid on n elements such that a subset is independent if and only if it has cardinality less than or equal to one.

An equivalent definition of M_X is as follows. Since $\Gamma \simeq G$, it is conjugate to a subgroup consisting of diagonal matrices whose diagonal entries are 1 or -1. So, we assume that Γ is such a subgroup. Let $\{\gamma_1, \ldots, \gamma_r\}$ be a generating set for Γ. Let M_Γ be the $\mathbb{Z}_2$–matrix such that $(M_\Gamma)_{ij} = \alpha^{-1}[(\gamma_i)_{jj}]$. Finally, define M_X to be the binary matroid represented by the column vectors of M_Γ.

Theorem 1. *The map $X \to M_X$ is a one–to–one correspondence between isometry classes of BSQ's and isomorphism classes of binary matroids.*

Proof. Let M be a binary matroid. Let $(M)_{ij}$ be a matrix which represents M over $\mathbb{Z}_2$. For each row R_i of $(M)_{ij}$ let γ_i be the diagonal matrix whose j^{th} diagonal entry is $\alpha(M_{ij})$. Let Γ be the group generated by the γ's. The unique representability of binary matroids ([4]) implies that any other binary matrix which represents M can be obtained from $(M)_{ij}$ by elementary row operations and column reordering. Elementary row operations do not change Γ. Reordering the columns of $(M)_{ij}$ gives a group conjugate to Γ. So the isometry class of $X_M = S^{n-1}/\Gamma$ is well defined. By construction $M \to X_M$ and $X \to M_X$ are inverse correspondences.

Remark 1. When $G = (\mathbb{Z}_p)^r, p$ an odd prime, M_X is constructed by giving the non–trivial representations in $FG_\mathbb{R}$ their natural independence relations and then adding one loop for each copy of the trivial representation. If $p = 3$, then the analog of Theorem 1 still holds [16]. However, for all other primes the correspondence is no longer a bijection. All $\mathbb{Z}_p$–lens spaces of dimension $2n-1$ have $U_{1,n}$ as their associated matroid, but for $p \geq 5$ and $n \geq 2$ there are pairs of $\mathbb{Z}_p$–lens spaces of dimension $2n-1$ which are not even homotopy equivalent [7].

Theorem 1 is used in [16] to construct a "dictionary" between the language of binary matroids and the geometry and topology of BSQ's. Most matroid concepts, including independence, basis, spanning, circuit, cocircuit, rank, closure, hyperplane, direct sum, parallel sum, two–sum, and connectivity have easily described analogs in terms of the geometry of BSQ's. Using this dictionary it is possible to translate almost any property of binary matroids into a geometric property of BSQ's. Theorems 2, 3, and 4 below are a small sample of what is possible.

In order to use Theorem 1 effectively we use the definition of M_X in terms of M_Γ. Notice that the elements of M_X are the columns of M_Γ and that the i^{th}–column corresponds to the action of Γ on e_i, the i^{th}–unit coordinate vector in $\mathbb{R}^n$. When both e_i and $-e_i$ are under discussion they will be denoted by e_i^+ and e_i^- respectively. We will also designate the elements of M_X by $E = \{e_1, \dots, e_n\}$, relying on context to distinguish the elements of M_X from the unit coordinate vectors. In order to facilitate going back and forth between M_X and X_M we introduce the following notation.

Definition 1. *Let $A \subseteq E$. The "A–sphere" is*

$$S^A \equiv \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_i = 0 \textit{ whenever } e_i \notin A.\}$$

$$S^{A^\circ} \equiv \{(x_1, \dots, x_n) : x_i = 0 \textit{ if and only if } e_i \notin A.\}$$

The image of S^A in X is denoted $\bar{A}$. Similarly, the image of S^{A° in X is $\bar{A}^\circ$. We abbreviate $\overline{\{e_i\}}$ by $\bar{e}_i$.

Proposition 2. *Let X be a BSQ. Then $e_i \in M_X$ is a loop of M_X if and only if $X = \varSigma X_{M/e_i}$.*

Proof. Suppose e_i is a loop of M_X. Then the i^{th}–column of $M_\varGamma$ is all zeros. Hence e_i^+ and e_i^- are fixed points of $\varGamma$. Therefore $X = \varSigma(S^{E-e_i}/\varGamma)$. By definition $\varSigma(S^{E-e_i}/\varGamma) = \varSigma X_{M/e_i}$. The converse follows immediately from Theorem 1.

Proposition 3. *Let X be a BSQ. Then X has non–empty boundary if and only if M_X contains a coloop. Furthermore, e_i is a coloop of M_X if and only if X is isometric to CX_{M-e_i}.*

Proof. A BSQ has non–empty boundary if and only if the fixed point set for some $\gamma \in \varGamma$ has dimension one less than the dimension of X. This is equivalent to the existence of a row in the row space of $M_\varGamma$ consisting of a single 1 and the rest zeros. Hence $M_\varGamma$ is row equivalent to a matrix with a unit coordinate vector in the row space. This is exactly the condition for a representable matroid to contain a coloop.

Let e_i be a coloop of M_X. Then, since the row space of $M_\varGamma$ contains a unit coordinate vector, there is an element $\gamma \in \varGamma$ which leaves S^{E-e_i} fixed and interchanges e_i^+ with e_i^-. Hence, $X = S^{n-1}/(\varGamma_{e_i^+} \oplus < \gamma >)$ which is just $\varSigma X_{M-e_i}/ < \gamma >= CX_{M-e_i}$.

Clearly, $\bar{A}$ is isometric to $X_{M|A}$. So, the deletion $M - A$ corresponds to $\overline{E - A}$. The contraction M/A roughly corresponds to the space of directions at any point in $\bar{A}^\circ$.

Proposition 4. *Let $A \subseteq M_X$. Let $x \in \bar{A}^\circ$. Then the space of directions at x is isometric to $S^{|A|-2} * X_{M/A}$.*

Proof. The space of directions at x is S_x^{n-1}/Γ_x. Since $x \in \bar{A}^\circ$, any element of Γ_x also fixes S^A. Therefore Γ_x leaves $S_x^{n-1} \cap S^A$ fixed. Since $S_x^{n-1} \cap S^A$ is a sphere of dimension $|A| - 2$, the space of directions at x is $S^{|A|-2} * S^{E-A}/\Gamma_x$. An element γ in Γ is in Γ_x if and only if it fixes the i^{th}–coordinate for every $e_i \in A$. Hence, the corresponding row vectors in the row space of M_Γ are exactly those row vectors which have a zero in the i^{th}–column for every $e_i \in A$. Restricting this action to S^{E-A} is equivalent to removing the i^{th}–column for every e_i in A. Since this leaves precisely the row space of M/A, we see that $S^{E-A}/\Gamma_x = X_{M/A}$. (See [13, pg.112–113] for the matrix representation of M/A)

Corollary 1. *Let* $e_i \in M_X$*. Then* $S_{\bar{e}_i} = X_{M_X/e_i}$.

The structure theory of matroids begins with direct sum decomposition. Direct sum for binary matroids corresponds to join for BSQ's.

Proposition 5. *Let* M, M' *be binary matroids. Then* $X_{M\oplus M'}$ *is isometric to* $X_M * X_{M'}$.

Proof. Let $X_M = S^{n-1}/\Gamma$ and $X_{M'} = S^{m-1}/\Gamma'$. Then $X_{M\oplus M'}$ is isometric to $S^{n+m-1}/(\Gamma \oplus \Gamma')$, where Γ acts trivially on the last m coordinates and Γ' acts trivially on the first n coordinates. Thus,

$$X_{M\oplus M'} = S^{n+m-1}/(\Gamma \oplus \Gamma') = S^{n-1}/\Gamma * S^{m-1}/\Gamma'.$$

A matroid which cannot be written as the direct sum of two smaller matroids is called *connected* (also known as 2–connected). In [22] Whitney proved that every matroid has a unique direct sum decomposition into connected matroids. In combination with the above proposition we obtain the following theorem.

Theorem 2. *Every BSQ has a unique (up to order) decomposition,*

$$X = X_1 * X_2 * \cdots * X_k$$

such that each X_i *is not the join of other BSQ's.*

When X is not contractible the homological computations in the next two sections allow us to determine the number of factors in the above decomposition from the volume of X and its topological connectivity. Since Γ acts freely on all but a subset of measure zero, $r(M_X) = \log_2(vol\, S^{n-1}/vol\, X)$.

Theorem 3. *Let* X *be a non–contractible binary sperical quotient. Let* r *equal* $\log_2(vol\ S^{n-1}/vol\ X)$*. Suppose* X *is* k*–connected but not* $k+1$*–connected. Then the number of factors in the above decomposition of* X *is* $k - r + 2$.

Proof. We first check that a BSQ is non–contractible if and only if M_X contains no coloops. One direction is Proposition 3. Conversely, suppose than M_X contains no coloops. Then $T(M_X;0,y) = y^{n-r}+$ lower degree terms [3]. By Theorem 6 in the next section, $H_{n-1}(X;\mathbb{Z}_2) = \mathbb{Z}_2$, so X is not contractible.

The fundamental group of X is trivial unless $X = \mathbb{R}P^{n-1}$ [11]. So $\tilde{H}_{k+1}(X,\mathbb{Z})$ is the first non–trivial reduced homology group of X. The number of components in the direct sum decomposition of M_X is equal to the first i such that $b_{0,i}$ is non–zero [3]. According to Theorem 7 the first non–trivial reduced homology group of X occurs in dimension $r-1+i$. So $k+1 = r-1+i$.

Corollary 2. *Let $r = \log_2(vol\ S^{n-1}/vol\ X)$. Then X is at least $r-1$ connected.*

The following theorem gives a test for determining whether or not a quotient of a smooth manifold by an elementary 2–group is homeomorphic to a manifold. (It would also apply to any group all of whose isotropy subgroups are elementary 2–groups.) The equivalence of (2) and (3) is contained in [11, Theorem 2.14].

Theorem 4. *Let X be a BSQ. Then the following are equivalent:*

1. *M_X is a direct sum of circuits.*
2. *X is homeomorphic to a sphere.*
3. *X has the homology of a sphere.*

Proof. $1 \to 2$. Let $D(\Delta^{n-1})$ be the double of the spherical $n-1$ simplex. This is the space obtained by taking two copies of the spherical $n-1$ simplex and identifying them along their boundaries. For $n \geq 2, D(\Delta^{n-1})$ is homeomorphic to a sphere, and from [11] we know that $D(\Delta^{n-1})$ is a BSQ. The volume of $D(\Delta^{n-1})$ tells us that the rank of $M_{D(\Delta^{n-1})}$ is $n-1$. The only matroid of rank $n-1$, cardinality n (i.e. $|E| = n$) and no coloops is the n–circuit. Hence the BSQ corresponding to a circuit of size 2 or more is homeomorphic to a sphere. The circuit of cardinality one is a loop, and its corresponding BSQ is S^0. Thus, if M_X is a direct sum of circuits, X is homeomorphic to a join of spheres.

$2 \to 3$ is trivial.

$3 \to 1$. Suppose X is a homology sphere. By Theorem 7, $T(M_X;0,y)$ equals y^{n-r}. Hence $M_X = M_1 \oplus \cdots \oplus M_{n-r}$ and $T(M_i;0,y) = y$ for each M_i [8]. Therefore, the cardinality of each M_i is one more than its rank. Since M_X does not contain a coloop (otherwise $T(M_X;0,y) = 0$), none of the M_i contains a coloop. As noted above, the only matroids with these properties are circuits.

It is possible to realize M_X concretely in X without resorting to $\varGamma$. Instead of using elements of $G_{\mathbb{R}}$ for E, we can use the points which are the images of the their corresponding invariant subspaces. Let l be the number of loops in M_X. Then it is possible to find pairs of points, $LP_X = \{\{x_1, y_1\}, \ldots, \{x_l, y_l\}\}$ such that $d(x_i, y_i) = \pi$ and the distance between points from distinct pairs is $\pi/2$. These pairs represent the loops of M_X. Let X' be the set of all points in X which are exactly $\pi/2$ from all points in LP_X. Then X' is a space of maximal extent of dimension $n-l-1$. As such it contains at least one (there may be infinitely many) collection of $n-l$ points $E_{X'} = \{x_{l+1}, \ldots, x_n\}$ which are mutually $\pi/2$ apart. These represent the non–loop elements of M_X. The union of LP_X and $E_{X'}$ is the image of the unit coordinate vectors of S^{n-1}. If A is a subset of $E_{X'}$, then the convex hull of A in X (or X' since X' is totally convex in X) is just the image in X of the corresponding sphere in S^{n-1}. The matroid properties of $E_{X'}$, and hence M_X, can now be recovered using any of the translations. For instance, if $A \subseteq E_{X'}$, then the rank of A can be read off from the volume formula above. Another example would be: A is a circuit if and only if the convex hull of A is isometric to a double simplex.

6 $\varDelta_\star(M)$

Here we introduce $\varDelta_\star(M)$. It is a chain complex associated to *any* matroid. When M is a binary matroid it coincides with $\tilde{\varDelta}_\star(X_M)$ which will be defined in Sect. 9. The calculation of $H_\star(\varDelta_\star(M))$ will provide the machinery needed to determine $H_\star(X_M; \mathbb{Z}_2)$. Finally, we compare $\varDelta_\star(M)$ to other complexes involving matroids with similar homological properties. In this section M is any matroid and we work with integer coefficients.

Definition 2. *$\varDelta_m(M)$ is the free abelian group on the subsets of E of cardinality $m+1$. For a subset A of E we denote its corresponding generator by $[A]$.*

In order to define the boundary map, fix an ordering $\{e_1, \ldots, e_n\}$ of E. Let $e_i \bullet A$ stand for e_i is a coloop of A. The boundary map is defined by

$$\partial([A]) = \sum_{e_{i_k} \bullet A} (-1)^k [A - e_{i_k}], \text{ where } A = \{e_{i_0}, \ldots, e_{i_m}\}.$$

The usual proof that $\partial^2 = 0$ in a simplicial complex can be adapted to this situation. It is sufficient to note that if e is a coloop of A, then f is a coloop of $A - e$ if and only if f is a coloop of A.

Since $\varDelta_\star(M - e_n)$ is a subcomplex of $\varDelta_\star(M)$, there is a short exact sequence of complexes,

$$0 \longrightarrow \Delta_\star(M-e_n) \xrightarrow{in_\star} \Delta_\star(M) \xrightarrow{j_\star} \frac{\Delta_\star(M)}{\Delta_\star(M-e_n)} \longrightarrow 0 \qquad (1)$$

Let $A \subseteq M$ and let $\widehat{[A]}$ be the coset of $[A]$ in $\Delta_\star(M)/\Delta_\star(M-e_n)$. Let ψ be the map from $\Delta_\star(M)/\Delta_\star(M-e_n)$ to $\Delta_{\star-1}(M/e_n)$ defined by $\psi(\widehat{[A]}) = 0$ if $e_n \notin A$, and $\psi\widehat{[A]} = [A-e_n]$ if $e_n \in A$.

Proposition 6. *As defined above, ψ is a chain isomorphism of degree -1.*

Proof. Since ψ maps a basis to a basis it is a group isomorphism. To see that it is a chain map, it is sufficient to note that for any e and f in A, e is a coloop of A if and only if e is a coloop of A/f.

Substituting $\Delta_\star(M/e_n)$ in (1) we obtain a long exact sequence,

$$\cdots \xrightarrow{\partial_m} H_m(M-e_n) \xrightarrow{in_m} H_m(M) \xrightarrow{j_m} H_{m-1}(M/e_n) \xrightarrow{\partial_{m-1}} \ldots \qquad (2)$$

Proposition 7. *If e_n is a coloop of M, then $H_\star(M) \equiv 0$.*

Proof. Since e_n is a coloop we can identify M/e_n with $M-e_n$. Let z be a cycle in $H_m(M/e_n), z = \sum_k c_k[A_k], c_k \in \mathbb{Z}, A_k \subseteq M/e_n$. Define $z \cup e_n$ to be $\sum_k c_k[A_k \cup e_n]$ in $\Delta_m(M)$. Then $z \cup e_n$ satisfies $j_{m+1}(z \cup e_n) = z$ and $\partial^M_{m+1}(z \cup e_n) = z$. Hence, in $H_m(M/e_n), \partial_m(z) = z$ in $H_m(M-e_n)$ and ∂_m is an isomorphism.

In order to analyze (2) when e_n is not a coloop we introduce the following subgroups of $\Delta_\star(M)$.

Definition 3. $\Delta^s_m(M) =< \{[A] : |A| = m+1$ *and* $r(A) = s\} >$.

By definition $\partial(\Delta^s_m(M)) \subseteq \Delta^{s-1}_m(M)$. So, in a rank r matroid a non–zero element of $\Delta^r_m(M)$ cannot be a boundary. In addition, an element of $\Delta_m(M)$ is a cycle if and only if its projection onto each $\Delta^s_m(M)$ is a cycle. (Note: ∂ is the boundary map of the complex $\Delta_\star(M)$, not the map in (2).)

Proposition 8. *Let M be a matroid of rank r. If $e_n \in M$ is not a coloop of M, then j_m is a surjection in (2). Furthermore, H_m is a finitely generated free abelian group, $H_m(M) \simeq H_m(M-e_n) \oplus H_{m-1}(M/e_n)$ and every cycle in $H_m(M)$ has a representative in $\Delta^r_m(M)$.*

Proof. The induction begins with $n = 2$. The three possibilities, $U_{0,2}, U_{1,2}$ and $U_{0,1} \oplus U_{1,1}$ (deleteing and contracting on the loop) are easy to check.

Assume the proposition holds for all matroids of cardinality $n-1$. Since e_n is not a coloop of $M, r(M-e_n) = r(M) = r$. By the induction hypothesis every element of $H_m(M-e_n)$ has a representative in $\Delta^r_m(M-e_n)$.

Therefore, no non–zero member of $H_m(M - e_n)$ can be a boundary in $H_m(M)$. Hence in_m is injective and j_m is surjective. Since $H_m(M - e_n)$ and $H_{m-1}(M/e)$ are finitely generated free abelian groups, $H_m(M)$ is also such a group and $H_m(M) \simeq H_m(M - e_n) \oplus H_{m-1}(M/e_n)$. To show that $H_m(M)$ has the required representatives, we show that a basis of $H_m(M)$ does. Let $\mathbf{B}$ be a basis of $H_m(M-e_n)$ with representatives in $\Delta^r_m(M-e_n)$, and let $\mathbf{B}'$ be a basis of $H_{m-1}(M/e_n)$ with representatives in $\Delta^{r-1}_{m-1}(M/e_n)$. Let $j^{-1}_m(\mathbf{B}')$ be any set of preimages of $\mathbf{B}'$. Then $in_m(\mathbf{B}) \cup \mathbf{B}'$ is a basis for $H_m(M)$. Clearly $in_m(\mathbf{B})$ has the requisite representatives. Suppose $b' = \sum_k c_k[A_k]$ is in $\mathbf{B}'$. Then,

$$j^{-1}_m(b') = \sum_k c_k[A_k \cup e_n] + \sum_{k'} c_{k'}[A_{k'}],$$

where e_n is not contained in any A_k or $A_{k'}$. In M, $A_k \cup e_n$ has rank r. If we remove all the terms from the right hand side which are in $\Delta^s_m(M)$, $s < r$ we are left with a cycle in $\Delta^r_m(M)$ whose image under j is still b'.

Let $|\chi|(M^\star;t)$ be the polynomial whose coefficients are the absolute value of the coefficients of the characteristic polynomial of the dual of M. Equivalently, $|\chi|(M^\star;t) = T(M;0,t+1)$.

Theorem 5. *Let M be a rank r matroid of cardinality n. Let $\mathbb{P}_M(t) = \sum_{m=-1}^{n-1}$ rank $(H_m(M))t^m$ be the Poincaré polynomial of $H_\star(M)$. Then, $\mathbb{P}_M(t) = t^{r-1}|\chi|(M^\star;t)$.*

Proof. Let $g_M(t) = \mathbb{P}_M(t)/t^{r-1}$. We verify that $g_M(t)$ satisfies Tutte recursion for $T(M;0,t+1)$. If M consists of a single coloop, then $\mathbb{P}_M(t) = 0$, so $g_M(t) = 0$. If M consists of a single loop, then $\mathbb{P}_M(t) = 1 + 1/t$, so $g_M(t) = t+1$. If e_n is a coloop of M, then $H_\star(M) = 0$, so $g_M(t)$ is also zero. If e_n is a loop of M, then $M - e_n = M/e_n$ and the rank of each is r. By the above proposition and induction $g_M(t) = (t+1)g_{M/e_n}(t)$. Finally, if e_n is neither a loop nor a coloop of M, then another application of Proposition 8 and induction show that $g_M(t) = g_{M-e_n}(t) + g_{M/e_n}(t)$.

Remark 2. Since $H_\star(M)$ is a finitely generated free abelian group all of the above computations hold with any coefficient group.

The first chain complex with Betti numbers equal to the coefficients of the characteristic polynomial of an associated matroid was found by Baclawski [1]. Other complexes whose Poincaré polynomials are related to the characteristic polynomial of matroids are Whitney homology [5] and the complexes arising from the homology and cohomology of the complements of complex hyperplane arrangements [12]. In [24] Yuzvinsky constructs differential graded algebras for a class of partially ordered sets which include geometric lattices (matroids without loops or circuits of cardinality two). If we ignore the multiplicative structure, the result is a chain complex closely related to $\Delta_\star(M)$. Since the construction makes sense for any matroid we do not restrict it to geometric lattices. The grading is slightly different from [24] so that it matches $\Delta_\star(M)$.

Let $C_m(M)$ be the free abelian group on the subsets of E of cardinality $m+1$. Let $A = \{e_{i_o}, \ldots, e_{i_k}, \ldots, e_{i_m}\}$. Define the boundary map by,

$$\partial([A]) = \sum_{e_{i_k} \not\bullet A} (-1)^k [A - e_{i_k}].$$

As shown in [24], $(C_\star(M), \partial)$ is a chain complex whose Poincaré polynomial is $t^r|\chi|(M; 1/t)$. To see the relationship between $C_\star(M)$ and $\Delta_\star(M)$ we apply three "dual" operations. First, rename the basis elements with their complement in E and change the dimension of the chain groups to match the size of their new labels. Call this new cochain complex $C^\star(M)$. For instance, what was $C_{-1} = \mathbb{Z}[\emptyset]$, becomes $C^\star_{n-1} = \mathbb{Z}[E]$. Now apply Hom in the usual way. Designate the new chain complex $C^{\star\star}(M)$ and label the basis dual to the basis of $C^\star$ with the same labels. Finally, we use matroid duality. Using the fact that an element e is not a coloop of a subset A of M if and only if e is a coloop of $(E - A) \cup e$ in $M^\star$, it is easy to show that $C^{\star\star}(M)$ and $\Delta_\star(M^\star)$ are almost identical. The only difference is that in $C^{\star\star}$ the sign of the boundary operator is determined by using an element's position in the complement, instead of its position in the subset.

7 Homology of BSQ's

In this section we compute $H_\star(X, \mathbb{Z}_2)$ when X is a BSQ. If M_X has a loop or a coloop, then it is easy to compute the homology groups of X using either Proposition 2 or Proposition 3. In order to compute the homology groups of X when e_i is neither a loop nor a coloop we use the mapping cone description in [11].

Let e_i be an element of M_X which is neither a loop nor a coloop. Then Γ can be written as $\Gamma_{e_i^+} \oplus \gamma$, where γ interchanges e_i^+ and e_i^- and acts non–trivially on S^{E-e_i}. Corollary 1 implies that $S^{n-1}/\Gamma_{e_i^+} = \Sigma S_{\bar{e}_i}$. Now, γ interchanges the upper and lower cones of $\Sigma S_{\bar{e}_i}$. In addition, γ also acts on the middle copy of $S_{\bar{e}_i}$ in $\Sigma S_{\bar{e}_i}$. Thus, X is the mapping cone of the quotient map $f : S_{\bar{e}_i} \to S_{\bar{e}_i}/ < \gamma >$. Let $D_{\bar{e}_i} = S^{E-e_i}/\Gamma = X_{M-e_i}$. The Mayer-Vietoris sequence associated to this mapping cone construction is,

$$\cdots \xrightarrow{f_m} \tilde{H}_m(D_{\bar{e}_i}, \mathbb{Z}_2) \xrightarrow{in_m} \tilde{H}_m(X, \mathbb{Z}_2) \xrightarrow{\partial_m} \tilde{H}_{m-1}(S_{\bar{e}_i}, \mathbb{Z}_2) \xrightarrow{f_{m-1}} \ldots \quad (3)$$

Proposition 9. *If e_i is neither a loop nor a coloop of M_X, then ∂_m is surjective.*

Proof. See Sect. 9.

Theorem 6. *Let $X = S^{n-1}/\Gamma$ be a BSQ with $\Gamma \simeq (\mathbb{Z}_2)^r$.*
Let $\tilde{\mathbb{P}}_X(t) = \sum_{m=0}^{n-1} dim_{\mathbb{Z}_2} \tilde{H}_m(X, \mathbb{Z}_2)t^m$. *Then* $\tilde{\mathbb{P}}_X(t) = t^{r-1}T(M_X; 0, t)$.

Proof. Let M be a rank r binary matroid and define $g_M(t) = \tilde{\mathbb{P}}_{X_M}(t)/t^{r-1}$. In order to prove that $g_M(t) = T(M; 0, t)$ it is sufficient to show that $g_M(t)$ satisfies Tutte recursion. When M is a loop X_M is two points and when M is a coloop X is a single point. In both cases $g_M(t) = T(M; 0, t)$. If e_i is a coloop then X_M is a cone and both $g_M(t)$ and $T(M; 0, t)$ are zero. If e_i is a loop then the rank of M and M/e_i are the same. Since X_M is $\Sigma X_{M/e_i}, g_M(t) = tg_{M/e_i}(t)$. Finally, assume that e_i is neither a loop nor a coloop of M. Then $r(M/e_i) = r(M) - 1$ and $r(M - e_i) = r(M)$. Combining this with (3) and the surjectivity of $\partial_\star$ we see that $g_M(t)$ is equal to $g_{M-e_i}(t) + g_{M/e_i}(t)$.

Remark 3. As pointed out in Remark 1, when $G = (\mathbb{Z}_p)^r$ the corresponding matroid does not determine the quotient space unless $p = 3$. However, the homology groups of the quotient space are determined by the associated matroid and there is a formula analogous to the one above [16].

8 Integral coefficients and matroid applications

In this section we compute $H_\star(X, \mathbb{Z})$ and the (reduced) Euler characteristic of a BSQ in terms of invariants of M_X. This will point us toward new inequalities for the Tutte polynomial of a matroid representable over *any* finite field. In addition, we will discover a surprising connection between $\mu(M)$ and whether or not it is affine. A matroid representable over F is *affine* over F if it has a representative set of vectors which are in the complement of a linear hyperplane.

A binary matroid is *Eulerian* if for some (any) matrix which represents M over $\mathbb{Z}_2$ every row of the matrix has an even number of ones. See [3] for several other equivalent definitions and the relationship between Eulerian matroids and Eulerian circuits in graphs.

Proposition 10. *Let $X = S^{n-1}/\Gamma$ be a BSQ. Then,*

$$\tilde{H}_{n-1}(X, \mathbb{Z}) = \begin{cases} \mathbb{Z}, & if\ M_X\ is\ Eulerian. \\ 0, & otherwise. \end{cases}$$

Proof. By [2] $H_{n-1}(X,\mathbb{Q})$ is isomorphic to the subgroup of $H_{n-1}(S^{n-1},\mathbb{Q})$ left fixed by the action of Γ. Thus, $H_{n-1}(X,\mathbb{Q})$ is $\mathbb{Q}$ if all the involutions in Γ are orientation preserving, zero otherwise. Since X is an $n-1$ dimensional CW–complex (see Sect. 9), the same statement holds with $\mathbb{Q}$ replaced with $\mathbb{Z}$. Now the proposition follows from the fact that each element of Γ is orientation preserving if and only if each row of M_Γ has an even number of ones.

In order to compute $H_m(X,\mathbb{Z})$ when $m < n-1$, we use $j_\star$, the coefficient homomorphism associated to the short exact sequence of groups,

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{j} \mathbb{Z}_2 \longrightarrow 0 \tag{4}$$

Proposition 11. *Let* X *be a BSQ of dimension* $n-1$. *Then for all* m *less than* $n-1,\ j_m : \tilde{H}_m(X,\mathbb{Z}) \to \tilde{H}_m(X,\mathbb{Z}_2)$ *is injective.*

Proof. Induction on n. When n is one all relevant homology groups are zero. If M_X contains a coloop, then X is contractible and all homology groups are zero. If M_X contains a loop, then X is the suspension of an lower dimensional BSQ, and since $j_\star$ behaves nicely with suspension, there is no problem. Suppose e_n is neither a loop nor a coloop of M_X. Consider the following commutative diagram.

$$\begin{tikzcd}
{} \arrow[r, "f_m"] & \tilde{H}_m(D_{\bar{e}_n},\mathbb{Z}) \arrow[r, "in_m"] \arrow[d, "j_m^{D_{\bar{e}_n}}"'] & \tilde{H}_m(X,\mathbb{Z}) \arrow[r, "\partial_{m-1}"] \arrow[d, "j_m^X"'] & \tilde{H}_{m-1}(S_{\bar{e}_n},\mathbb{Z}) \arrow[r, "f_{m-1}"] \arrow[d, "j_{m-1}^{S_{\bar{e}_n}}"'] & {} \\
{} \arrow[r, "f_m"] & \tilde{H}_m(D_{\bar{e}_n},\mathbb{Z}_2) \arrow[r, "in_m"] & \tilde{H}_m(X,\mathbb{Z}_2) \arrow[r, "\partial_{m-1}"] & \tilde{H}_{m-1}(S_{\bar{e}_n},\mathbb{Z}_2) \arrow[r, "f_{m-1}"] & {}
\end{tikzcd}$$

On the bottom row f_m and f_{m-1} are zero by Proposition 9. Now suppose $m \le n-3$. Then $j_m^{D_{\bar{e}_n}}$ and $j_{m-1}^{S_{\bar{e}_n}}$ are injective by the induction hypothesis. Therefore $in_m \circ j_m^{D_{\bar{e}_n}}$ is injective. Hence in_m on the top line is also injective. By the snake lemma j_m^X is injective. Finally, suppose $m = n-2$. If M_X is $U_{0,n-1}$ then $X = S^{n-1}$ and the homology groups in question are trivial. If M_X is $U_{1,2k+1}$, then $X = \mathbb{R}P^{2k}$. In this case, $\tilde{H}_m(X,\mathbb{Z}) \simeq \tilde{H}_m(X,\mathbb{Z}_2) \simeq \mathbb{Z}_2$ and j_{m-2} is an isomorphism, so the proposition holds. For all other binary matroids it is possible to find some e_i in M_X such that $M_X - e_i$ is not Eulerian. So, by reordering if necessary, we can assume that $\tilde{H}_{n-2}(D_{\bar{e}_n},\mathbb{Z})$ is $\{0\}$ and argue as before.

In order to describe $H_\star(X;\mathbb{Z})$ we introduce the following fragment of $T(M;t,0)$.

Definition 4. *Let* M *be any matroid.*

$$b(M;m,t) \equiv (-1)^m[b_{m,0}\, t^m + b_{m-1,0}\, t^{m-1} + \cdots + b_{1,0}\, t].$$

Note that $b(M;m,-1)$ is just an alternating sum of the coefficients of $T(M;t,0)$ of degree less than or equal to m.

Theorem 7. *Let X be a BSQ of dimension $n-1$ with $r(M_X)=r$. If $0 \leq m \leq n-2$, then*

$$\tilde{H}_m(X,\mathbb{Z}) = \begin{cases} (\mathbb{Z}_2)^{b(M_X^\star;m-r+1,-1)}, & if\ r \leq m \leq n-2 \\ 0, & if\ m < r. \end{cases}$$

Proof. By Proposition 11 and the long exact sequence associated to (4), multiplication by 2 on $\tilde{H}_m(X,\mathbb{Z})$ is the zero map. Hence $\tilde{H}_m(X,\mathbb{Z})$ is an elementary 2–group and $\tilde{H}_m(X,\mathbb{Z}_2) \cong \tilde{H}_m(X,\mathbb{Z}) \oplus \tilde{H}_{m-1}(X,\mathbb{Z})$. Theorem 6 and induction finish the proof.

An immediate consequence of the above theorem is that for any binary matroid M and all $1 \leq m \leq n-r, b(M^\star;m,-1) \geq 0$. By duality, this is equivalent to the non–negativity of $b(M;m,-1)$ whenever $1 \leq m \leq r$ and M is binary. When $m=r, b(M;m,-1) = (-1)^r T(-1,0) = \chi(M;2)$. In [10] Crapo and Rota show that for any matroid representable over a finite field $F, \chi(M;|F|^k) \geq 0$. Thus the non–negativity of $b(M;m,-1)$ can be viewed as a generalization of the Crap–Rota inequality when $|F|=2$ and $k=1$. In fact, the inequality holds for any finite F and any $k>0$.

Theorem 8. *Let M be a matroid representable over a finite field F and let $k>0$. Then for any $1 \leq m \leq r(M), b(M;m,1-|F|^k) \geq 0$.*

Proof. Induction on $n=|E|$. If $n=1$ then M is either a loop or a coloop. If the former, then there is nothing to show. If M consists of a single coloop, then $b(M;1,1-|F|^k) = |F|^k - 1 > 0$.

Now assume that the theorem holds for all matroids of cardinality $n-1$. If e_n is a loop, then $b(M;m,1-|F|^k)=0$ since $T(M;x,0)=0$. If e_n is a coloop, then $b(M;m,1-|F|^k)$ is just $b(M-e_n;m-1,1-|F|^k)$ which is non–negative by the induction hypothesis. Finally, suppose e_n is neither a loop nor a coloop. Then for $m<r, b(M;m,1-|F|^k)$ is non–negative by Tutte recursion and the induction hypothesis. When $m=r$ we are back to the Crapo–Rota inequality.

The $\mathbb{Z}_2$–dimension of the integer homology groups of X_M give an interpretation of the above inequality when $|F|=2$ and $k=1$.

Problem 1. Find interpretations for the inequalities in the above theorem when $|F|>2$ and $k>1$.

Using rational coefficients, Theorems 10 and 7 we see that the reduced Euler characteristic of a BSQ X is ± 1 if M_X is Eulerian, 0 otherwise. From

Theorem 6 the reduced Euler characteristic of X can also be written as the alternating sum $\pm(b_{0,n-r} - b_{0,n-r-1} \cdots + (-1)^{n-r+1} b_{0,1})$. On the other hand, $|\mu(M^\star_X)|$ is the sum $b_{0,n-r} + \cdots + b_{0,1}$. Hence, $\mu(M^\star_X)$ is odd if and only if M_X is Eulerian. A binary matroid is Eulerian if and only if its dual is affine [3]. So, a binary matroid is affine if and only if its Möbius invaritant is odd. One direction of this statement generalizes to matroids representable over any finite field.

Theorem 9. *Let M be a matroid representable over a finite field F. If $\mu(M) \neq 0 \; mod \; (|F|)$, then M is affine over F. If $|F| = 2$, then the converse also holds.*

Proof. Assume that M is not affine over F. Then by the Crapo–Rota critical problem formula ([10]) $\chi(M; |F|) = 0$. Since $\mu(M)$ is the constant term of $\chi(M; t)$, reducing both sides of this equation modulo $|F|$ shows that $\mu(M) = 0 \; mod \; (|F|)$.

For any field other that $\mathbb{F}_2$ there are counterexamples to the converse of the above theorem. The matroid $U_{|F|,|F|+1}$ is affine over any field of cardinality three or greater and $\mu(U_{|F|,|F|+1}) = |F|$.

9 Proof of Proposition 9

Binary spherical quotients have a natural CW structure inherited from a Γ invariant simplicial complex on S^{n-1}. The vertices of the simplicial complex on the sphere are $V = \{e_1^{\pm}, \ldots, e_n^{\pm}\}$. The higher dimensional simplices are the spherical simplexes whose vertices are any subset of V which contains only one of e_i^+, e_i^- for each i. This complex is invariant under the action of Γ and induces a CW–structure on X. We denote the associated CW–chain complex by $\Delta_\star(X)$. For any subset A of M_X, $\bar{A}$ is a subcomplex of X.

We denote the CW chain which is the sum of all cells of dimension $|A| - 1$ contained in $\bar{A}$ by $\tilde{A}$. We rely on context to determine whether a simplex is in S^{n-1} or represents a cell in X. Throughout this section *we use $\mathbb{Z}_2$ coefficients* and use M for M_X.

Definition 5. *$\tilde{\Delta}(X)$ is the subgroup of $\Delta_\star(X)$ generated by $\{\tilde{A} : A \subseteq E.\}$*

As we will see below, $\tilde{\Delta}(X)$ is a subcomplex of $\Delta(X)$ such that the inclusion map is a surjection in homology. Furthermore, $\tilde{\Delta}(X)$ provides preimages for ∂_m in Proposition 9.

Let B be a basis for A. Since B is a basis for A, each cell in $\tilde{A}$ has a unique representative cell in S^{n-1} containing e_i^+ for each $e_i \in B$. Hence, a set of representative cells for $\tilde{A}$ can be formed by using e_i^+ for each e_i in B, and all of the $2^{|A|-|B|}$ combinations of $+'$s and $-'$s for the e_j in A but not in B.

Example 4. Let $\varGamma$ be the group generated by

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Then $M_\varGamma = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$ and $B = \{e_1, e_2\}$ is a basis of $A = \{e_1, e_2, e_3, e_4\}$. One set of representative cells for $\tilde{A}$ is

$$\{(e_1^+, e_2^+, e_3^+, e_4^+), (e_1^+, e_2^+, e_3^+, e_4^-), (e_1^+, e_2^+, e_3^-, e_4^+), (e_1^+, e_2^+, e_3^-, e_4^-)\}.$$

Proposition 12. *Let $A \subseteq M$. Then,*

$$\partial(\tilde{A}) = \sum_{e_i \bullet A} \widetilde{A - e_i}$$

Proof. Suppose e_i is a coloop of A. When e_i is removed from each representative cell we are left with a set of representatives for the cells of $\widetilde{A - e_i}$. If e_i is not a coloop of A, then choose a basis B of A which does not contain e_i and a set of representatives for $\tilde{A}$ as above. When e_i is removed in the boundary operation we are left with two copies of a set of representatives for $\widetilde{A - e_i}$, one which used to contain e_i^+ and one which used to contain e_i^-. Thus, these cells make no contribution to $\partial(\tilde{A})$.

An immediate consequence of the above proposition is that $\tilde{\varDelta}(X)$ can be canonically identified with $\varDelta_\star(M)$ with $\mathbb{Z}_2$ coefficients. Similarly, $\tilde{\varDelta}(D_{\bar{e}_i})$ and $\tilde{\varDelta}(S_{\bar{e}_i})$ can be identified with $\varDelta(M - e_i)$ and $\varDelta(M/e_i)$ respectively (also $\mathbb{Z}_2$ coefficients). We can also easily identify f_m in (3) with ∂_m in (2), and ∂_m in (3) with j_m in (2). Thus, we obtain the long exact sequence,

$$\cdots \xrightarrow{f_m} \tilde{H}_m(\tilde{\varDelta}(X_{M-e_i})) \xrightarrow{in_m} \tilde{H}_m(\tilde{\varDelta}(X)) \xrightarrow{\partial_m} \tilde{H}_{m-1}(\tilde{\varDelta}(X_{M/e_i})) \xrightarrow{f_{m-1}} \cdots \tag{5}$$

Proposition 13. *Suppose that e_i is neither a loop nor a coloop in M. Then ∂_m is surjective for all m in the long exact sequence (5).*

Proof. Proposition 8.

The proof of Proposition 9 is now simple. For convenience we restate the proposition.

Proposition 14. *If e_i is neither a loop nor a coloop of M, then ∂_m is surjective in the long exact sequence,*

$$\cdots \xrightarrow{f_m} \tilde{H}_m(D_{\bar{e}_i}, \mathbb{Z}_2) \xrightarrow{in_m} \tilde{H}_m(X, \mathbb{Z}_2) \xrightarrow{\partial_m} \tilde{H}_{m-1}(S_{\bar{e}_i}, \mathbb{Z}_2) \xrightarrow{f_{m-1}} \cdots$$

Proof. First identify $D_{\bar{e}_i}$ with X_{M-e_i}, and $S_{\bar{e}_i}$ with X_{M/e_i}. For each space $Y \in \{D_{\bar{e}_i}, X, S_{\bar{e}_i}\}$, let ϕ denote the inclusion $\tilde{\Delta}_\star(Y) \to \Delta_\star(Y)$. The proof is by induction on n with the additional hypothesis that $\phi_\star$ is always a surjection (even if e_i is a loop or a coloop of M). Consider the following commutative diagram.

$$\begin{tikzcd}
\dots \arrow[r, "f_m"] & \tilde{H}_m(\tilde{\Delta}(D_{\bar{e}_i})) \arrow[r, "in_m"] \arrow[d, "\phi_\star"] & \tilde{H}_m(\tilde{\Delta}(X)) \arrow[r, "\partial_m"] \arrow[d, "\phi_\star"] & \tilde{H}_{m-1}(\tilde{\Delta}(S_{\bar{e}_i})) \arrow[r, "f_{m-1}"] \arrow[d, "\phi_\star"] & \dots \\
\dots \arrow[r, "f_m"] & \tilde{H}_m(D_{\bar{e}_i}) \arrow[r, "in_m"] & \tilde{H}_m(X) \arrow[r, "\partial_m"] & \tilde{H}_{m-1}(S_{\bar{e}_i}) \arrow[r, "f_{m-1}"] & \dots
\end{tikzcd}$$

If n is one then M is either a loop or a coloop. So we only need to check that $\phi_\star$ is surjective. If M is a coloop then X is a point and all (reduced) homology groups are trivial. When M is a loop, X is two points and the only non–trivial homology group is $H_0 \simeq \mathbb{Z}_2$. In this case it is easy to see that $\tilde{H}_0(\tilde{\Delta}) \simeq \mathbb{Z}_2$ and that ϕ is an isomorphism.

Now assume that the proposition and the extra induction hypothesis hold for $n-1$. By Proposition 13 and the induction hypothesis $\phi_{m-1} \circ \partial_m$ is surjective. So, in the lower exact sequence ∂_m is surjective, in_m is injective and f_{m-1} is zero. The induction hypothesis and the snake lemma show that $\phi_\star$ is surjective on $\tilde{H}_\star(\tilde{\Delta})$.

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