Problem 0.1. Verify that the following differential equation \((x + \frac{3}{2}y)dx + (\frac{3}{2}x + y)dy = 0\) is exact. Then solve it.

Proof Clearly,

\[
\frac{\partial}{\partial y}(x + \frac{3}{2}y) = \frac{\partial}{\partial x}(\frac{3}{2}x + y) = \frac{3}{2}
\]

and so our differential equation is exact.

Then, one writes \(F = \int (x + \frac{3}{2}y)dx = \frac{x^2}{2} + \frac{3xy}{2} + g(y)\) and then \(F_y = \frac{3x}{2} + g'(y)\). Since this last expression has to be equal to \(N = \frac{3}{2}x + y\), it follows that \(g'(y) = y\) which means \(g(y) = \frac{y^2}{2}\). As a consequence the implicit solution takes the form

\[
\frac{x^2}{2} + \frac{3xy}{2} + \frac{y^2}{2} = C.
\]

Problem 0.2. Find the explicit solution for the initial value problem \(2\sqrt{x} \frac{dy}{dx} = \cos^2 y, y(16) = \frac{\pi}{4}\).

Proof After separating the variables, one gets

\[
\int \sec^2 y dy = \int \frac{dx}{2\sqrt{x}}
\]

and so \(\tan y = \sqrt{x} + C\), which means that \(y(x) = \tan^{-1}(\sqrt{x} + C)\). Now, since \(y(16) = \frac{\pi}{4}\) it follows that \(C = -3\) and so the final answer is \(y(x) = \tan^{-1}(\sqrt{x} - 3)\)

Problem 0.3. Solve the initial value problem \(x^3 \frac{dy}{dx} + 2x^2y = \sin x, y(1) = 5\) on the positive real axis.

Proof One can rewrite it as

\[
\frac{dy}{dx} + \frac{2}{x}y = \frac{\sin x}{x^3}.
\]

In this case \(P(x) = \frac{2}{x}\) and so \(\bar{P}(x) = \int P(x)dx = 2 \ln x = \ln x^2\). This means that we have to multiply the equation with the factor \(e^{\bar{P}(x)} = x^2\). We thus obtain the new equation

\[
x^2 \frac{dy}{dx} + 2xy = \frac{\sin x}{x}
\]

which means \([x^2y(x)]' = \frac{\sin x}{x}\) and so \(x^2y(x) = \int^x \frac{\sin t}{t}dt + C\) and since \(y(1) = 5\) it follows that \(C = 5\). Then, the solution therefore becomes \(y(x) = \frac{1}{x^2}(5 + \int^x \frac{\sin t}{t}dt)\).
Problem 0.4. A recipient contains 500 liters (L) of 50 kg of salt dissolved in water. Pure water is pumped into the recipient at a rate of 10 L/s and the mixture (kept uniform by stirring) is pumped out at the same rate. How long it will take until only 5 kg of salt remains in the recipient?

Proof A careful look at the statement shows that the mixture differential equation is \( \frac{dx}{dt} = -x/50 \) with the initial condition \( x(0) = 50 \). Thus, the unique solution is \( x(t) = 50e^{-t/50} \). To find the time we are looking for we solve \( 50e^{-t/50} = 5 \) and so \( t = 50 \ln 10 \).

Problem 0.5. Suppose that an airplane departs from the point (50, 0) located 50 miles east of its intended destination, an airport located at the origin (0, 0). The plane travels with constant speed \( v_0 = 400 \) mi/h relative to the wind which is blowing due north with constant speed \( w = 30 \) mi/h. We also know that the plane’s pilot maintains its heading toward the origin. Calculate how far north does the wind blow the airplane?

Proof We are clearly in the “Flight trajectories” setting. In our particular case \( k = 1/10 \) and so the general trajectory solution (see page 64 in the book) takes the form

\[
y(x) = 25[(\frac{x}{50})^{9/10} - (\frac{x}{50})^{11/10}].
\]

To find the maximum of this function (this is what we are looking for!) we have to find that \( x_0 \) for which \( y'(x_0) = 0 \). A simple calculation shows that, one has to have

\[
\frac{9}{11} = (\frac{x_0}{50})^{2/10}.
\]

Using this, one obtains

\[
y(x_0) = 25[(\frac{9}{11})^{4.5} - (\frac{9}{11})^{5.5}]
\]

and this is the maximum number of miles with which the airplane is blown away.