1. Using the Method of Undetermined Coefficients and Variation of Parameters, find the particular solution of the following ODE: $2\ddot{x} + 8x = 3\sin 2t$.

2. The wavefunction of a quantum mechanical particle is determined by solving the Schrödinger Equation. In one spatial dimension, the Schrödinger Equation is:

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + U(x,t)\right]\Psi(x,t) = i\hbar \frac{\partial \Psi(x,t)}{\partial t}$$

where $\hbar$ is a constant, $m$ is the mass of the particle, $U(x,t)$ is the potential energy of the particle, $\Psi(x,t)$ is the wavefunction, and $i = \sqrt{-1}$.

(a) When $U = 0$, the Schrödinger Equation is mathematically equivalent to what partial differential equation?

(b) To solve the PDE for the wavefunction, assume $\Psi(x,t) = X(x)T(t)$, and show that this separation of variables leads to the following ODEs:

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + U\right)X = EX, \quad i\hbar \frac{dT}{dt} = ET$$

where $E$ is a positive constant (in fact, $E$ is the energy of the particle). In order to use this separation of variables, we need to make a restriction on $U$. What restriction is this?

(c) For the remainder of the problem, consider the potential energy function for a “particle in a box”:

$$U = \begin{cases} 0 & 0 < x < a \\ \infty & x < 0, x > a \end{cases}$$

Since the potential energy is infinite outside of the box, $\Psi(x,t) = 0$ for $x \leq 0$ and $x \geq a$. Solve for the wavefunction $\Psi(x,t)$ inside the box ($0 \leq x \leq a$) subject to the boundary conditions $\Psi(0,t) = \Psi(a,t) = 0$ and the general initial condition $\Psi(x,0) = f(x)$. Your answer should express $\Psi(x,t)$ as an infinite sum with coefficients in terms on an integral with $f(x)$ in the integrand. Can the energy $E$ of the terms in the sum take on any value?

(d) For $f(x) = bx(a - x)$, determine $\Psi(x,t)$. 

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3. Consider the following setup: You have a pendulum of mass $M$ and length $2L$. At the midpoint of the massless rod, the pendulum is attached to a horizontal spring of stiffness $k$ and a horizontal damper with a damping constant of $c$, see below.

(a) If $\theta$ is the counterclockwise angle from the vertical position, derive the equation of motion for pendulum in terms of $\theta$.

(b) If $L = 1m$, $M = 1kg$, $k = 1N/m$ and $c = 1N \cdot s/m$, what is the undamped frequency, and the damping ratio for this system ($g = 9.81m/s^2$).

Note: the undamped frequency and the damping ratio are the parameters $\omega_n$ and $\zeta$ from the simplified form of the equation of motion: $\ddot{\theta} + 2\zeta\omega_n \dot{\theta} + \omega_n^2 \theta = 0$.

(c) Keeping all constants the same, what should $c$ be changed to so that the system is critically damped?

4. Solve $\nabla^2 u = 0$ for the temperature $u(x, y)$ inside a rectangular slab. The right, left and top sides are insulated $u_x(0, y) = u_x(a, y) = u_x(x, b) = 0$, and the temperature distribution at the bottom is given by $u(x, 0) = f(x)$.

5. Find the equilibrium solutions as a function of $s$ for the following differential equation: $y' = y^2 - 2y + 1 + s$. Find the bifurcation point, draw the bifurcation diagram and label the stability of all the equilibrium solutions.